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A CHARACTERIZATION OF DUAL B^*-ALGEBRAS

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ABSTRACT. Let A be a B^*-algebra. The second conjugate space of A, denoted by A^{**}, is a B^*-algebra under the Arens multiplication. A new proof is given that A is a dual algebra if and only if the natural image of A in A^{**} is an ideal in A^{**}.

In this note we present an alternate proof to the following result of B. J. Tomiuk and Pak-Ken Wong [1].

THEOREM. *Let A be a (complex) B^*-algebra, A^{**} be its second conjugate space and π the canonical imbedding of A into A^{**}. Then A is a dual algebra if and only if $\pi(A)$ is an ideal in A^{**} with respect to the Arens multiplication.*

PROOF. For each $a \in A$ let L_a and R_a denote the operators on A defined by $L_a(x)=ax$ and $R_a(x)=xa$. If A^* denotes the conjugate space of A, let $L_a': A^* \to A^*$ denote the transpose of L_a defined by $L_a'(f)(x)=f(L_a(x))$ ($f \in A^*, x \in A$). The second transpose of L_a is then the mapping L_a'': $A^{**} \to A^{**}$ defined by $L_a''(F)(f)=F(L_a'(f))$ ($F \in A^{**}, f \in A^*$). Let R_a'' denote the second transpose of R_a defined similarly.

Then A is a dual algebra if and only if L_a and R_a are weakly compact operators for each a in A [2, p. 99]. By the generalized Gantmacher theorem [3, pp. 624–625], L_a and R_a ($a \in A$) are weakly compact if and only if $L_a''(A^{**}) \cup R_a''(A^{**}) \subseteq \pi(A)$. Let $*$ denote the Arens multiplication in A^{**}. It follows from the definition of this multiplication (see [1, p. 530]) that $L_a''(F)=\pi(a) * F$ and $R_a''(F)=F * \pi(a)$ for all $F \in A^{**}$, and therefore $L_a''(A^{**}) \cup R_a''(A^{**})=\pi(a) * A^{**} \cup A^{**} * \pi(a)$, which completes the proof.

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