SOME INTERDEPENDENCIES OF SUM AND PRODUCT INTEGRALS

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ABSTRACT. In a recent paper Davis and Chatfield show if $\int_a^b G^2 = 0$, then $\int_a^b G$ exists if and only if $\prod_a^b (1+G)$ exists and is not zero. In this paper we extend that result and prove if $\beta > 0$, $|G| < 1 - \beta$ on $[a, b]$ and $\int_a^b G^2$ exists, then $\int_a^b G$ exists if and only if $\prod_a^b (1+G)$ exists and is not zero. Furthermore, if $\prod_x^y (1+G) = \exp \int_x^y G$ for $a \leq x < y \leq b$, then $\int_a^b G^2 = 0$.

All integrals and definitions are of the subdivision-refinement type, R denotes the set of all real numbers, and functions are R -valued and understood to be defined only on $\{x, y\}$ in $R \times R$ such that $x < y$. The statement that G is bounded, $G \in OP^\circ$ or $G \in OB^\circ$ on $[a, b]$ means there exists a subdivision D of $[a, b]$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D , then

- (1) if $0 < q \leq n$, then $|G_q| < B$,
- (2) if $0 < u \leq v \leq n$, then $|\prod_{q=u}^v (1+G_q)| < B$, or
- (3) $\sum_{q=1}^n |G_q| < B$,

respectively, where $G_q = G(x_{q-1}, x_q)$. Similarly, statements of the form $G > b$ should be interpreted in terms of subdivisions and refinements. The sequence of functions $\{S_n\}_{n=1}^\infty$ converges uniformly to S on $[a, b]$ means if $\varepsilon > 0$ then there exists a subdivision D of $[a, b]$ and a positive integer N such that if $\{x_q\}_{q=0}^r$ is a refinement of D , $n > N$, and $0 < q \leq r$ then $|S_n(x_{q-1}, x_q) - S(x_{q-1}, x_q)| < \varepsilon$. Further, $G \in OL^\circ$ on $[a, b]$ if and only if $\lim_{x \rightarrow p^-} G(x, p)$, $\lim_{x \rightarrow p^+} G(p, x)$, $\lim_{x, y \rightarrow p^-} G(x, y)$ and $\lim_{x, y \rightarrow p^+} G(x, y)$ exist for $p \in [a, b]$. Also, $G \in OA^\circ$ or $G \in OM^\circ$ on $[a, b]$ if and only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$ or $\prod_x^y (1+G)$ exists for $a \leq x < y \leq b$ and $\int_a^b |1+G - \prod (1+G)| = 0$, respectively. For R -valued functions, the class OA° on $[a, b]$ is the class of functions G such that $\int_a^b G$ exists [4, p. 669]. The sources of these definitions are found in [2, p. 299] and [3, p. 493].

THEOREM 1. If $G \in OB^\circ$ on $[a, b]$, $\{S_n\}_{n=1}^\infty$ is a sequence of functions converging uniformly to S on $[a, b]$ and $\int_a^b S_n G$ exists for $n=1, 2, 3, \dots$, then $\int_a^b SG$ exists, $\lim_{n \rightarrow \infty} \int_a^b S_n G$ exists, and $\int_a^b SG = \lim_{n \rightarrow \infty} \int_a^b S_n G$.

The proof of Theorem 1 is omitted.

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LEMMA 1. If $\prod_a^b (1+G)$ exists and is not zero and $p \in (a, b]$, then there exists $\delta > 0$ such that if $p - \delta < x < p$, then $|G(x, p) - G(x, y) - G(y, p) - G(x, y)G(y, p)|$ is bounded as $y \rightarrow p^-$. If, furthermore, $G(y, p)$ is bounded as $y \rightarrow p^-$, then

$$\lim_{x, y \rightarrow p^-} [G(x, p) - G(x, y) - G(y, p) - G(x, y)G(y, p)] = 0.$$

LEMMA 2. If $\int_a^b G$ exists and $p \in (a, b]$, then

$$\lim_{x, y \rightarrow p^-} [G(x, y) + G(y, p) - G(x, p)] = 0.$$

Lemmas 1 and 2 can be established by using the Cauchy criteria for product and sum integrals, respectively.

THEOREM 2. If $\prod_a^b (1+G)$ exists and is not zero and $\int_a^b G^2$ exists, then $G \in OL^\circ$ on $[a, b]$.

PROOF. Since $\int_a^b G^2$ exists, it follows

(1) if $p \in (a, b]$, then $\lim_{x, y \rightarrow p^-} G(x, y) = 0$, and

(2) if $p \in [a, b)$, then $\lim_{x, y \rightarrow p^+} G(x, y) = 0$, and hence it is only necessary to show that

(3) if $p \in (a, b]$, then $\lim_{x \rightarrow p^-} G(x, p)$ exists, and

(4) if $p \in [a, b)$, then $\lim_{x \rightarrow p^+} G(p, x)$ exists.

Let $p \in (a, b]$. Thus, since (1) is true and $G(y, p)$ is bounded as $y \rightarrow p^-$, it follows from Lemma 1 that (3) is true. In a similar manner, statement (4) can be shown to be true; therefore, $G \in OL^\circ$ on $[a, b]$.

THEOREM 3. If $\int_a^b G$ exists and $\int_a^b G^2$ exists, then $G \in OL^\circ$ on $[a, b]$.

The proof is similar to the proof of Theorem 2.

THEOREM 4. If $-1 < G$ on $[a, b]$, then the following statements are equivalent:

(1) $\prod_a^b (1+G)$ exists and is not zero, and

(2) $\int_a^b \ln(1+G)$ exists.

Furthermore, $\prod_a^b (1+G) = \exp \int_a^b \ln(1+G)$.

INDICATION OF PROOF. (1) $\rightarrow$ (2).

$$\begin{aligned} \left| \ln \prod_a^b (1+G) - \sum_H \ln(1+G_q) \right| &= \left| \ln \left[\alpha \cdot \prod_H (1+G_q) \right] - \sum_H \ln(1+G_q) \right| \\ &= |\ln \alpha| \rightarrow 0. \end{aligned}$$

INDICATION OF PROOF. (2)→(1).

$$\begin{aligned} \left| \exp \int_a^b \ln(1+G) - \prod_H (1+G_q) \right| \\ = \left| \exp \left[\alpha + \sum_H \ln(1+G_q) \right] - \prod_H (1+G_q) \right| \\ = \left| (e^\alpha - 1) \prod_H (1+G_q) \right| \rightarrow 0. \end{aligned}$$

THEOREM 5. If $\beta > 0$, $|G| < 1 - \beta$ on $[a, b]$ and $\int_a^b G^2$ exists, then the following statements are equivalent:

- (1) $\prod_a^b (1+G)$ exists and is not zero, and
- (2) $\int_a^b G$ exists.

PROOF. (1)→(2). Note that $G^2 \in OA^\circ$ and OB° on $[a, b]$. It follows from Theorem 2 that $G \in OL^\circ$ on $[a, b]$. Hence, $\sum_{n=2}^p (-1)^{n-1} [G^{n-2}]/n \in OL^\circ$ on $[a, b]$ for $p=2, 3, 4, \dots$. Thus, $\int_a^b \{ \sum_{n=2}^p (-1)^{n-1} [G^{n-2}]/n \} G^2$ exists for $p=2, 3, 4, \dots$ [3, Theorem 2, p. 494]. Therefore, since

$$\left\{ \sum_{n=2}^p (-1)^{n-1} [G^{n-2}]/n \right\}_{p=2}^\infty$$

converges uniformly to $\sum_{n=2}^\infty (-1)^{n-1} [G^{n-2}]/n$ on $[a, b]$, it follows from Theorem 1 that

$$\int_a^b \left\{ \sum_{n=2}^\infty (-1)^{n-1} [G^{n-2}]/n \right\} G^2 = \int_a^b \sum_{n=2}^\infty (-1)^{n-1} G^n/n$$

exists. Further, Theorem 4 implies that

$$\int_a^b \ln(1+G) = \int_a^b \sum_{n=1}^\infty (-1)^{n-1} G^n/n$$

exists. Hence, $\int_a^b G$ exists.

PROOF. (2)→(1). By the use of Theorems 1 and 3, it follows as in the preceding paragraph that $\int_a^b \sum_{n=2}^\infty (-1)^{n-1} G^n/n$ exists. Thus,

$$\int_a^b \sum_{n=1}^\infty (-1)^{n-1} G^n/n = \int_a^b \ln(1+G)$$

exists. Therefore, Theorem 4 implies that $\prod_a^b (1+G)$ exists and is not zero.

THEOREM 6. If $\prod_a^b (1+G)$ exists and is not zero and $\int_a^b G$ exists, then G is bounded on $[a, b]$.

PROOF. We will show if $p \in (a, b]$ then $G(y, p)$ is bounded as $y \rightarrow p^-$ and $G(x, y)$ is bounded as $x, y \rightarrow p^-$ and if $p \in [a, b)$ then $G(p, y)$ is bounded as $y \rightarrow p^+$ and $G(x, y)$ is bounded as $x, y \rightarrow p^+$. Boundedness then follows by using the Heine-Borel theorem. Suppose there exists $p \in (a, b]$ such that $G(y, p)$ is unbounded as $y \rightarrow p^-$. Hence, there exists a sequence $\{y_q\}_{q=1}^\infty$ converging to p such that $y_q < y_{q+1} < p$ and $G(y_q, p) > q$. It follows from Lemmas 1 and 2 that there exist numbers x and B such that if $x < y_q < p$, then

- (1) $|G(x, p) - G(x, y_q) - G(y_q, p)| < B$, and
- (2) $|G(x, p) - G(x, y_q) - G(y_q, p) - G(x, y_q)G(y_q, p)| < B$.

Hence, $G(x, y_q)G(y_q, p)$ is bounded as $q \rightarrow \infty$. Thus, $\{G(x, y_q)\}_{q=1}^\infty$ must converge to zero, and as a result, $G(x, p) - G(x, y_q) - G(y_q, p)$ is unbounded as $q \rightarrow \infty$. This contradicts (1), and therefore, $G(y, p)$ is bounded as $y \rightarrow p^-$. Now, if $p \in (a, b]$, it follows from Lemma 2 that $G(x, y)$ is bounded as $x, y \rightarrow p^-$. Boundedness follows in a similar manner for the right limits of G .

Therefore, it follows from the Heine-Borel theorem that there exists a number B and a subdivision $\{x_q\}_{q=0}^n$ of $[a, b]$ such that if $x_{q-1} \leq x < y \leq x_q$ then $|G(x, y)| < B$.

LEMMA 3. If G is bounded on $[a, b]$ and $\int_a^b G$ exists, then there exists a number B such that if $a \leq x < y \leq b$ then $|\int_x^y G| < B$.

Since $G \in OA^\circ$ on $[a, b]$, the proof follows easily.

THEOREM 7. If $\prod_a^b (1+G)$ exists and is not zero, then the following statements are equivalent:

- (1) if $a \leq x < y \leq b$, then $\int_x^y G$ exists and $\prod_x^y (1+G) = \exp \int_x^y G$, and
- (2) $\int_a^b G^2 = 0$.

PROOF. Davis and Chatfield [1, Theorem 3, p. 744] show that (2) implies (1). Hence, we only need to show that (1) implies (2). We will do this by showing that

$$\int_a^b \left| \sum_{n=2}^{\infty} G^n/n! \right| = 0.$$

From this, the desired result follows.

Let $\varepsilon > 0$. Since G is bounded (Theorem 6), $\int_x^y G$ is bounded on $[a, b]$ (Lemma 3). Thus, $\prod_x^y (1+G)$ and $[\prod_x^y (1+G)]^{-1}$ are bounded for $a \leq x < y \leq b$. Therefore, $G \in OM^\circ$ on $[a, b]$ [2, Theorem 4.2, p. 305]. Also, $G \in OA^\circ$ on $[a, b]$. Thus, there exists a subdivision D of $[a, b]$ and a number B such that if $H = \{x_q\}_{q=0}^n$ is a refinement of D , then

- (1) $\sum_H |1 + G_q - \prod_{x_{q-1}}^{x_q} (1+G)| < \varepsilon/2$,
- (2) $|\exp G_q| < B$ for $x_{q-1}, x_q \in H$, and
- (3) $\sum_H |-1 + \exp[-G_q + \int_{x_{q-1}}^{x_q} G]| < \varepsilon/2B$.

Let $H = \{x_q\}_{q=0}^n$ be a refinement of D , then

$$\begin{aligned} \varepsilon/2 &> \sum_H \left| 1 + G_q - \prod_{x_{q-1}}^{x_q} (1 + G) \right| \\ &= \sum_H \left| 1 + G_q - \exp \int_{x_{q-1}}^{x_q} G \right| \\ &= \sum_H \left| 1 + G_q - \{\exp G_q\} \left\{ 1 - 1 + \exp \left[-G_q + \int_{x_{q-1}}^{x_q} G \right] \right\} \right| \\ &\geq \sum_H |1 + G_q - \exp G_q| - \sum_H |\exp G_q| \left| -1 + \exp \left[-G_q + \int_{x_{q-1}}^{x_q} G \right] \right| \\ &> \sum_H |1 + G_q - \exp G_q| - B\varepsilon/2B, \end{aligned}$$

and thus

$$\varepsilon > \sum_H |1 + G_q - \exp G_q| = \sum_H \left| \sum_{n=2}^{\infty} [G_q^n/n!] \right|.$$

Since $\int_a^b |e^G - 1 - G| = 0$ and $e^x - 1 - x$ has the minimum value of 0 at $x=0$,

$$\sum_{n=2}^{\infty} [G_q^{n-2}]/n! = \frac{1}{2} + \sum_{n=3}^{\infty} [G_q^{n-2}]/n! > \frac{1}{4}$$

for refinements of a suitable subdivision of $[a, b]$. Therefore, since

$$0 = \int_a^b \left| \sum_{n=2}^{\infty} G^n/n! \right| = \int_a^b |G^2| \left| \sum_{n=2}^{\infty} [G^{n-2}]/n! \right|,$$

it follows that $\int_a^b G^2 = 0$.

A special case of Theorem 7 is proved by J. A. Chatfield [Theorem 8 in "Equivalence of integrals," Proc. Amer. Math. Soc. (submitted)]. He shows that Theorem 7 is true for functions from $R \times R$ to the nonnegative numbers. However, his proof does not hold for functions from $R \times R$ to R .

Davis and Chatfield [1, p. 747] show that there exists a function G such that $\int_a^b G^2 = 0$, $\int_a^b G$ exists and $G \notin OB^\circ$ on $[a, b]$. In Theorem 8 we show that under slightly weaker conditions $G \in OP^\circ$. Note that $OB^\circ \subseteq OP^\circ$ [2, p. 299].

THEOREM 8. *If $\beta > 0$, $|G| < 1 - \beta$ on $[a, b]$, $\int_a^b G$ exists and $\int_a^b G^2$ exists, then $G \in OP^\circ$ on $[a, b]$.*

PROOF. It follows from Theorems 4 and 5 that $\int_a^b \ln(1+G)$ exists. Thus, since $\ln(1+G)$ is bounded on $[a, b]$, it follows from Lemma 3 that there

exists a subdivision D of $[a, b]$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D and $0 < u \leq v \leq n$, then $|\sum_{q=u}^v \ln(1 + G_q)| < B$. Hence, if $\{x_q\}_{q=0}^n$ is a refinement of D and $0 < u \leq v \leq n$, then

$$\begin{aligned} \left| \prod_{q=u}^v (1 + G_q) \right| &= \left| \exp \sum_{q=u}^v \ln(1 + G_q) \right| \\ &\leq \exp \left| \sum_{q=u}^v \ln(1 + G_q) \right| < e^B. \end{aligned}$$

Therefore, $G \in OP^\circ$ on $[a, b]$.

THEOREM 9. *If $\prod_a^b (1 + G)$ exists and is not zero and $\int_a^b G$ exists, then $G \in OL^\circ$ on $[a, b]$.*

PROOF. Let $p \in (a, b]$. Since G is bounded on $[a, b]$ (Theorem 6), it follows from Lemmas 1 and 2 that

- (1) $\lim_{x, y \rightarrow p^-} [G(x, y) + G(y, p) - G(x, p)] = 0$, and
- (2) $\lim_{x, y \rightarrow p^-} [G(x, p) - G(x, y) - G(y, p) - G(x, y)G(y, p)] = 0$.

Hence,

- (3) $\lim_{x, y \rightarrow p^-} G(x, y)G(y, p) = 0$.

Suppose $\lim_{y \rightarrow p^-} G(y, p)$ does not exist; then there exist a sequence $\{x_q\}_{q=1}^\infty$ converging to p and a positive number ε such that

- (4) $x_{q-1} < x_q < p$, and
- (5) $|G(x_{q-1}, p) - G(x_q, p)| > \varepsilon$.

There exists a subsequence $\{x_{q_i}\}_{i=1}^\infty$ of $\{x_q\}_{q=1}^\infty$ such that $\{G(x_{q_i}, p)\}_{i=1}^\infty$ is bounded away from zero. Hence, it follows from (3) that

$$\lim_{i \rightarrow \infty} G(x_{q_{i-1}}, x_{q_i}) = 0.$$

Thus,

$$\lim_{x, y \rightarrow p^-} [G(x, y) + G(y, p) - G(x, p)] \neq 0.$$

Since this contradicts (1), $\lim_{y \rightarrow p^-} G(y, p)$ exists. It now follows from (1) that $\lim_{x, y \rightarrow p^-} G(x, y) = 0$.

The existence of the right limits follows in a similar manner. Therefore, $G \in OL^\circ$ on $[a, b]$.

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