MEAN GROWTH OF INNER FUNCTIONS

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ABSTRACT. Let A be a closed subset of the closed unit disc. It is shown that there exists a "universal growth function" $\varphi(r, A)$ such that $1 - \int_0^{2\pi} |q(re^{ix})|^2 dx/2\pi = O(\varphi(r, A))$ for all inner functions $q(z)$ whose zeroes lie in $A \cap \{|z| < 1\}$ and whose singularities in the unit circle lie on $A \cap \{|z| = 1\}$, if and only if the Lebesgue measure of $A \cap \{|z| = 1\}$ is zero.

The H^2 Hardy class of functions in the unit disc $D = \{z: |z| < 1\}$ of the complex plane is, by definition, the set of all functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$, analytic on D , satisfying

$$\limsup(r \rightarrow 1) M(f, r) < \infty,$$

where $M(f, r) = \int_0^{2\pi} |f(re^{ix})|^2 dm = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$, $0 \leq r < 1$ ($dm = (1/2\pi) dx$ is the normalized Lebesgue measure on ∂D).

It is clear that $M(f, r)$ is convex and nondecreasing with r and that $M(f, r)$ converges to $M(f) = \sum_{n=0}^{\infty} |c_n|^2 < \infty$, as $r \rightarrow 1$. However, by taking suitable lacunary series, it can be easily seen that there is no function $\varphi(r)$ such that

(1) $\varphi(r)$ is positive and decreasing for $r \in [0, 1)$, and $\lim(r \rightarrow 1)\varphi(r) = 0$, and $M(f) - M(f, r) = O(\varphi(r))$, for all $f \in H^2$.

The problem that we want to analyze here is this:

Is it possible to find a function $\varphi(r)$, satisfying (1) and such that

$$(2) \quad 1 - M(q, r) = O(\varphi(r))$$

holds for every inner function $q(z)$?

We recall that $q \in H^2$ is inner if it has radial limits $q(e^{ix}) = \lim(r \rightarrow 1) q(re^{ix})$ of modulus one a.e. (dm) (for the existence of these limits and basic properties of H^2 functions and inner functions, see [2]).

Our first result provides an affirmative answer for a particular class of inner functions. The examples following the theorem will show that the

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result is sharp; in particular, there is no function $\varphi(r)$ satisfying (1) such that (2) holds for every inner function. These results are contained in the author's thesis [1].

The canonical expression of an inner function $q(z)$ is (see [2]) $q(z) = \lambda b(z)s(z)$, where $\lambda \in \partial D$,

$$b(z) = \prod_k (\bar{a}_k/|a_k|) \cdot (a_k - z)/(1 - \bar{a}_k z) \quad (= \text{Blaschke product of } q)$$

with $a_k \in D$ for all k and $\sum_k (1 - |a_k|) < \infty$ ($\bar{a}_k/|a_k| = 1$, if $a_k = 0$) and

$$s(z) = e(z, \mu) = \exp \left\{ - \int_0^{2\pi} (e^{it} + z)/(e^{it} - z) d\mu(t) \right\}$$

(=singular part of q), where μ is a finite nonnegative Borel measure on ∂D , singular with respect to dm . We shall define

$$\text{Sp}(q) = \text{supp}(\mu) \cup \text{closure}\{a_k\}$$

($q(z)$ can be continued analytically to $z = \alpha$, for some α in ∂D , if and only if $\alpha \notin \text{Sp}(q)$).

THEOREM. *Let A be a closed subset of the closed unit disc D^- , such that $A \cap \partial D$ has (linear) Lebesgue measure zero. Then there exists a function $\varphi(r, A)$ satisfying (1) such that (2) holds for every inner function $q(z)$ with $\text{Sp}(q) \subset A$.*

We shall need an auxiliary result.

LEMMA. *Let p, q be two inner functions and assume that $1 - M(p, r) \leq \alpha^2$, $1 - M(q, r) \leq \beta^2$, $0 \leq \alpha, \beta < 1$ (for some r , $0 \leq r < 1$). Then $1 - M(pq, r) \leq 3(\alpha + \beta)$.*

PROOF. Let $K(p, \alpha, r) = \{e^{ix} \in \partial D : |p(re^{ix})| > 1 - \alpha\}$ and let $K(q, \beta, r)$ be similarly defined. Since $|p(z)| \leq 1$, the inequality $1 - M(p, r) \leq \alpha^2$ and a straightforward computation show that the measure of $K(p, \alpha, r)$ must be greater than $1 - \alpha$. Similarly, $m(K(q, \beta, r)) > 1 - \beta$ and therefore, if

$$N(pq, r) = \{e^{ix} \in \partial D : |p(re^{ix})q(re^{ix})| > (1 - \alpha)(1 - \beta)\},$$

then $m(N(pq, r)) > (1 - \alpha)(1 - \beta)$ (observe that $N(pq, r)$ contains $K(p, \alpha, r) \cap K(q, \beta, r)$), whence the result follows. Q.E.D.

Using this lemma, it will be enough to prove the Theorem for Blaschke products and for singular inner functions separately.

(a) **BLASCHKE PRODUCTS.** Consider the function

$$b(z, a) = (\bar{a}/|a|) \cdot (a - z)/(1 - \bar{a}z), \quad a = re^{it} \in D,$$

and let ε , $0 < \varepsilon < 1$, be given; then the set $\{z: |b(z, a)| < 1 - \varepsilon(1 - r)\}$ is a disc having the rectilinear segment $[-r_i e^{it}; r_s e^{it}]$, where $r_i = (1 - \varepsilon)/(1 + \varepsilon r)$ and $r_s = (1 - \varepsilon + r + \varepsilon r)/(1 + r - \varepsilon r + \varepsilon r^2)$, as a diameter. We have $r_i < 1 - \varepsilon$, $r_s < 1 - \varepsilon(1 - r)^2$.

Let A be as in the Theorem and let $b(z) = \prod_k b(z, a_k)$ be a Blaschke product such that $a_k \in A \cap D$ for all k , and $\sum_k (1 - |a_k|) \leq M < \infty$. We are going to construct a function $\alpha(r, A, M)^2$ satisfying the conditions (1), such that $1 - M(b, r) \leq \alpha(r, A, M)^2$, for all Blaschke products satisfying the above conditions.

For each ε , $0 < \varepsilon < 1$, define

$$A_\varepsilon = \bigcup \{z: |b(z, a)| < 1 - \varepsilon(1 - |a|); a \in A \cap D\},$$

and let $D_{\varepsilon, t}$ be the disc having the rectilinear segment

$$[-(1 - \varepsilon)e^{it}; (1 - \varepsilon(1 - r_i)^2)e^{it}],$$

where $r_i = \max\{r: re^{it} \in A\}$ ($0 \leq t \leq 2\pi$), as a diameter. The above estimate shows that

$$A \cap D \subset A_\varepsilon \subset A_\varepsilon^\sim = \text{closure} \bigcup \{D_{\varepsilon, t}: 0 \leq t \leq 2\pi\}.$$

It follows that $A \subset \bigcap \{A_\varepsilon^\sim: 0 < \varepsilon < 1\}$ and $A \cap \partial D = A_\varepsilon^\sim \cap \partial D$, for all ε . Furthermore, if $A_{\varepsilon, r} = \{e^{it}: re^{it} \in A_\varepsilon^\sim\}$, then

$$\lim(r \rightarrow 1) m(A_{\varepsilon, r}) \leq m(A \cap \partial D) = 0.$$

Now, if $z \in D \setminus A_\varepsilon^\sim$, then

$$|b(z)| = \prod_k |b(z, a_k)| \geq \prod_k [1 - \varepsilon(1 - |a_k|)] \geq C(\varepsilon, M),$$

where $C(\varepsilon, M) \rightarrow 1$ as $\varepsilon \rightarrow 0$ for fixed M (namely, if A does not intersect the disc of radius $r_0 > 0$ about the origin, then $C(\varepsilon, M)$ can be taken equal to $\exp\{-\varepsilon M/r_0\}$).

Fix ε and choose r so that $m(A_{\varepsilon, r}) < \varepsilon$. Then

$$M(b, r) = \int_0^{2\pi} |b(re^{ix})|^2 dm \geq C(\varepsilon, M)^2(1 - \varepsilon) \rightarrow 1, \quad \text{as } \varepsilon \rightarrow 0.$$

It is clear that the relation between ε and r can be inverted in order to obtain a function $\alpha(r, A, M)^2$, which depends only on M and the shape of A , such that $1 - M(b, r) \leq \alpha(r, A, M)^2 \rightarrow 0$, as $r \rightarrow 1$, independently of the distribution of the zeroes $\{a_k\}$ in A .

(b) SINGULAR FUNCTIONS. Now consider the function $s(z) = e(z, \mu)$, where the measure μ is supported by $A \cap \partial D$. If $z = re^{ix}$, then

$$|s(z)| = \exp\left\{-\int_0^{2\pi} P(r, x - t) d\mu(t)\right\},$$

where $P(r, x-t) = (1-r^2)/(1+r^2-2r \cos(x-t))$ is the Poisson kernel of the disc. This expression shows that the infimum of $|s(z)|$ over all functions $s(z)$ satisfying the above conditions and such that $\|\mu\| \leq M$ is actually attained by taking $\mu = M\delta_{c(z)}$, where $\delta_{c(z)}$ is the Dirac measure placed at one of the points $c(z)$ such that

$$|z - c(z)| = \min\{|z - e^{it}| : e^{it} \in A\} \quad (|c(z)| = 1).$$

Now observe that the set $\{z : |d(z; 0, M)| \leq 1 - \varepsilon\}$ (where $d(z; t, N) = \exp\{N(z + e^{it})/(z - e^{it})\}$) is the closed disc having the rectilinear segment $[-r_{\varepsilon, M}; 1]$ where $r_{\varepsilon, M}^2 = [M + \log(1 - \varepsilon)]/[M - \log(1 - \varepsilon)]$, as a diameter.

Let $D_{\varepsilon, M, t}$ be the closed disc having the rectilinear segment $[-r_{\varepsilon, M}e^{it}; e^{it}]$ as a diameter, and define

$$B_{\varepsilon, M} = \bigcup \{D_{\varepsilon, M, t} : e^{it} \in A\}.$$

We have $A \cap \partial D = \bigcap \{B_{\varepsilon, M} : 0 < \varepsilon < 1\} = B_{\varepsilon, M} \cap \partial D$ for all ε , and

$$\lim(r \rightarrow 1) m(\{e^{it} : re^{it} \in B_{\varepsilon, M}\}) = m(A \cap \partial D) = 0.$$

Now we can proceed as in Case (a) to obtain a function $\beta(r, A, M)^2$ satisfying the conditions (1), which only depends on M and the shape of A , such that $1 - M(s, r) \leq \beta(r, A, M)^2$, independently of the mass distribution of the measure μ on $A \cap \partial D$.

From (a), (b) and the Lemma, it follows that

$$1 - M(q, r) = 1 - M(bs, r) \leq 3[\alpha(r, A, M) + \beta(r, A, M)] = \varphi(r, A, M),$$

for any inner function $q(z)$ such that $\text{Sp}(q) \subset A$ and $\sum_k (1 - |a_k|) \leq M$, $\|\mu\| \leq M$.

Finally, if $\varphi(r, A)$ is any function satisfying (1) such that

$$\lim(r \rightarrow 1) \varphi(r, A, M)/\varphi(r, A) = 0,$$

for all $M > 0$ (such a function can be easily constructed by standard arguments), then $1 - M(q, r) = O(\varphi(r, A))$, for all $q(z)$ with $\text{Sp}(q) \subset A$.

The proof of the Theorem is complete now.

EXAMPLE. Let $\varphi(r)$ satisfy (1). There exist a Blaschke product $b(z)$ and a sequence of radii $0 < r_1 < \dots < r_n < \dots < 1$ converging to 1, such that $1 - M(b, r_n) > n \cdot \varphi(r_n)$, for all n . Furthermore, $b(z)$ can be constructed so that $\text{Sp}(b) \cap \partial D = \{1\}$.

The construction is simple. Let $A_n = \{e^{it} : |t| < \pi/2^n\}$, and let

$$f_n(z) = \exp\left\{-\int_{-\pi/2^n}^{\pi/2^n} (e^{it} + z)/(e^{it} - z) dm(t)\right\}.$$

Then $|f_n(z)| < 1$ on D and

$$\begin{aligned} |f_n(e^{ix})| &= e^{-1} && \text{if } e^{ix} \in A_n \\ &= 1 && \text{if } e^{ix} \in \partial D \setminus A_n \end{aligned} \quad (\text{a.e. } dm).$$

Construct the sequence $\{r_n\}$ by induction, so that $n \cdot \varphi(r_n) < 2^{-n-4}$. Let $-\pi/2^n = y_0 < y_1 < \cdots < y_N = \pi/2^n$ be a sufficiently small partition of A_n so that $y_j - y_{j-1} = 2\pi/2^n N$, $j=1, \dots, N$, and

$$\left| f_n(z) - \prod_{j=1}^N d(z; y_j, 1/2^n N) \right| < 4^{-n-5}, \quad \text{for } |z| \leq r_n.$$

Approximate each of the functions $d(z; y_j, 1/2^n N)$ by a Blaschke product whose zeros cluster at e^{iy_j} (this can be done by using a classical result of O. Frostman, see [2]) and then approximate each of these Blaschke products by a suitable partial finite product on the closed disc of radius r_n . Thus we can obtain a *finite Blaschke product* $b_n(z)$ such that

- (1) $|f_n(z) - b_n(z)| < 4^{-n-5}$, for $|z| \leq r_n$, and
- (2) $b_n(re^{ix}) \neq 0$, if $r < r_n$ or $e^{ix} \notin A_{n-1}$.

Define $b(z) = \prod_{n=1}^{\infty} b_n(z)$; we have

$$b(0) = \prod_n b_n(0) = \prod_n f_n(0)[b_n(0)/f_n(0)] > e^{-1} \prod_n (1 - 4^{-n-5}) > 0.$$

Therefore, $b(z)$ is actually a Blaschke product (see [2]), and

$$\begin{aligned} 1 - M(b, r_n) &\geq 1 - M(b_n, r_n) \\ &\geq 1 - M(f_n, r_n) + [M(f_n, r_n) - M(b_n, r_n)] \\ &> 1 - M(f_n) - 2^{-n-4} > 2^{-n-3} - 2^{-n-4} > n \cdot \varphi(r_n). \end{aligned}$$

It is apparent that $z=1$ is the only cluster point for the zeroes of $b(z)$. It is also clear that a similar method can be used to prove that, if $C \subset \partial D$ is any closed subset of positive Lebesgue measure, then there exists a singular inner function $s(z) = e(z, \mu)$ such that $\text{supp}(\mu) \subset C$ and, for a suitable increasing sequence of radii converging to 1, $1 - M(s, r_n) > n \cdot \varphi(r_n)$, for all n .

A FINAL REMARK. The referee called our attention to the following result: If $s(z)$ is a (nonconstant) *singular* inner function, then $1 - M(s, r)$ cannot converge to zero "too fast" (as $r \rightarrow 1$). In fact, it was shown in [4] that $s(z)$ is subordinated to $d(z; 0, R)$ (for some $R > 0$); now we can apply the corollary on p. 165 of [3] to show that $M(s, r) \leq M(d, r)$, from which we can deduce that

$$1 - M(s, r) \geq 1 - M(d, r) \geq c(1 - r)^{1/2} \quad (\text{for some } c > 0);$$

i.e.,

$$1 - M(s, r) \neq o((1 - r)^{1/2}).$$

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