ON THE FRATTINI SUBGROUPS OF CERTAIN GENERALIZED FREE PRODUCTS OF GROUPS¹

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ABSTRACT. Let $G = (\prod_{i \in I}^* A_i)_H$ be the generalized free product of the groups A_i amalgamating the subgroup H . We show that if G is residually finite and the groups A_i have compatible H -filters then the Frattini subgroup $\Phi(G)$ is contained in the maximal G -normal subgroup in H . If the groups A_i are free and H is finitely generated of infinite index in one A_i then $\Phi(G) = 1$. We also show that if H is simple then $\Phi(G) = 1$ or H^G .

1. The purpose of this paper is to continue our effort to answer the questions raised by Higman and Neumann [5]: whether the Frattini subgroup of a generalized free product can be larger than the amalgamated subgroup and whether such groups necessarily have maximal subgroups. In [8] Whittemore answered these questions for generalized free products of finitely many free groups amalgamating a cyclic subgroup and for generalized free products of finitely many finitely generated abelian groups. In [2] Djoković and Tang showed that if G is the generalized free product amalgamating the subgroup H which satisfies the minimal condition on subgroups then the Frattini subgroup $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H . In this paper we shall extend the results in [2] and [8]. In particular, we show that if $G = (\prod_{i \in I}^* A_i)_H$ is residually finite and the A_i have compatible H -filters then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H . In the case of $G = (\prod_{i \in I}^* A_i)_H$ where each A_i is free and H is finitely generated then $\Phi(G) = 1$ if H is of infinite index in one of the A_i . We also show that if the amalgamated subgroup is simple then $\Phi(G) = 1$ or H^G .

The notation used is in general standard:

$G = (A * B)_H$ denotes the generalized free product of A and B amalgamating the subgroup H .

A subgroup N is said to be G -normal if $N \triangleleft G$.

$G \setminus H$ denotes the set of elements of G not contained in H .

$\langle a_1, \dots, a_n \rangle$ denotes the group generated by the elements $a_1, \dots, a_n$.

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Let $x \in G = (\prod_{i \in I}^* A_i)_H$. Then *init* x and *end* x shall respectively denote an initial letter and an end letter of x when x is expressed as $a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n}$ where $a_{\alpha_i} \in A_j$ and no two consecutive a_{α_i} belong to the same A_j . We note that for a given x both *init* x and *end* x are not uniquely defined. However any two *init* x belong to the same A_i for a given x . The same is true for *end* x . Also $l(x)$ shall denote the length of x in G .

The following terms and notation are the same as Baumslag [1]:

$\{S_i; i \in I\}$ is a filter of G if each S_i is a normal subgroup of finite index in G and $\bigcap_I S_i = 1$.

$\{S_i; i \in I\}$ is called an H -filter of G where H is a subgroup of G if it is a filter of G and $\bigcap_I S_i H = H$.

If $G = (A * B)_H$ and $\{A_i; i \in I\}$, $\{B_i; i \in I\}$ are filters of A and B respectively then they are said to be compatible with respect to H if $A_i \cap H = B_i \cap H$ for $i \in I$.

The following result from [2] will be needed:

THEOREM 1.1. *Let $G = (A * B)_H$. If H satisfies the minimal condition on subgroups then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H .*

REMARK. Throughout the paper $G = (A * B)_H$ shall be assumed to be a nontrivial generalized free product. Although we prove our results for $G = (A * B)_H$ only, it is not difficult to extend the results to $G = (\prod_{i \in I}^* A_i)_H$ for arbitrary I .

2. We shall first prove a theorem which we need in §3.

THEOREM 2.1. *Let $G = (A * B)_H$. If $\Phi(G) \cap H = 1$ then $\Phi(G) = 1$.*

PROOF. Let $x \in \Phi(G)$ and $h \in H$. If $h^x \in H$ then $[x, h] \in \Phi(G) \cap H = 1$. Thus x and h commute whenever $h^x \in H$. Suppose $\Phi(G) \neq 1$. Since $\Phi(G) \cap H = 1$ and $\Phi(G) \triangleleft G$, it is not difficult to see that there exists an element $c \in \Phi(G)$ such that *init* $c \in B \setminus H$ and *end* $c \in A \setminus H$; for example, if $x \in \Phi(G)$ is of the form $a_1 b_1 \cdots a_n b_n a_{n+1}$ where $a_i, b_i \notin H$, then for $b \in B \setminus H$, $c = x^b \cdot x \in \Phi(G)$ will be of the required form. Clearly the order of c is infinite, whence $c^2 \neq 1$. Let $X = \langle A^{c^2}, B \rangle$. We shall show that $c \notin X$. Suppose $c \in X$. Then c would be of the form:

$$(1) \quad c = b_0 c^{-2} a_1 c^2 b_1 c^{-2} a_2 c^2 b_2 \cdots c^{-2} a_n c^2 b_n$$

where $a_i \in A$ and $b_i \in B$. Without loss of generality we can assume n to be minimal. We shall show that equation (1) cannot hold.

Case 1. $a_1 \in H$ and $a_1^c \in H$. As noted earlier this means $a_1^c = a_1 \in H$. Thus $b_1 \notin H$, for otherwise n would not be minimal. Also $a_2^c \notin H$; for if $a_2^c \in H$ then we must have $a_2 \in H$ since *init* $c \in B \setminus H$. But then $[c, a_2] = 1$

which would violate the minimal choice of n . Using the same argument we can assume $b_i^{c^{-1}} \notin H$ for $i=2, \dots, n-1$ and $a_i^c \notin H$ for $i=2, \dots, n$. Now it is easy to see that for $n > 1$ the length of the R.H.S. (right hand side) of (1) is greater than $l(c)$ since $\text{init } c \in B \setminus H$ and $\text{end } c \in A \setminus H$. Hence (1) cannot hold. On the other hand if $n \leq 1$, then the length of the R.H.S. of (1) is 0 or 1. But by the choice of c , $l(c) > 1$. Hence again (1) cannot hold.

Case 2. $a_1 \in H$ and $a_1^c \notin H$. As in Case 1, because of the minimal choice of n , $b_i^{c^{-1}} \notin H$ for $i=1, \dots, n-1$ and $a_i^c \notin H$ for $i=2, \dots, n$ if $n \geq 2$. It is not difficult to see that for $n \geq 1$ the length of the R.H.S. of (1) is always greater than $l(c)$. On the other hand if $n=0$ then the length of the R.H.S. of (1) is always less than $l(c)$. Hence (1) cannot hold.

Case 3. $a_1 \in A \setminus H$. Since $\text{init } c \in B \setminus H$, it follows that $a_1^c \notin H$. Applying the same argument for Case 2 we conclude that (1) cannot hold.

Thus in all possible cases the equation (1) cannot be true. Therefore $c \notin X$, whence $X \subsetneq G$. On the other hand $G = \langle X, c \rangle$. This implies that $c \notin \Phi(G)$ contradicting the assumption that $c \in \Phi(G)$. This contradiction comes from the assumption that $\Phi(G) \neq 1$. Hence $\Phi(G) = 1$.

Since $\Phi(G) \cap H$ is normal in H we have the following corollary:

COROLLARY 2.2. *Let $G = (A * B)_H$. If H is a simple group then $\Phi(G) = 1$ or H^G .*

3. We shall now proceed to show that if $G = (A * B)_H$ is residually finite and A and B have compatible H -filters then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H .

LEMMA 3.1. *Let $G = (A * B)_H$. Let S be a subgroup of A and $T = S^G$, the normal closure of S in G . Then $G/T = (\bar{A} * \bar{B})_{\bar{H}}$ where $\bar{A} = AT/T$, $\bar{B} = BT/T$ and $\bar{H} = HT/T$.*

PROOF. Let $\bar{G} = (\bar{A} * \bar{B})_{\bar{H}}$. Let $\theta_A: A \rightarrow \bar{A}$ and $\theta_B: B \rightarrow \bar{B}$ be respectively the restrictions of the canonical homomorphism from G onto G/T . Then by a well-known theorem of B. H. Neumann [7, p. 505] θ_A and θ_B can be extended to a homomorphism θ from G onto $\bar{G}$. Now it is not difficult to show that $\ker \theta$ is T . Thus G/T is not only isomorphic but actually equal to $\bar{G}$.

LEMMA 3.2. *Let $G = (A * B)_H$. If $U \triangleleft A$ and $V \triangleleft B$ such that $U \cap H = V \cap H$ then $U^G \cap A = U$ and $V^G \cap B = V$.*

PROOF. Clearly $U \subseteq U^G \cap A$. Let $x \in U^G \cap A$. Then $x = u^g$ for some $u \in U$ and $g \in G$. Let $g = a_1 b_1 \cdots a_n b_n$ where $a_i \in A$, $a_i \in A \setminus H$ for $i=2, \dots, n$ and $b_n \in B$, $b_i \in B \setminus H$ for $i=1, \dots, n-1$. Then

$$(2) \quad x = b_n^{-1} a_n^{-1} \cdots b_1^{-1} a_1^{-1} u a_1 b_1 \cdots a_n b_n.$$

Since $x \in A$ we have $l(x) \leq 1$. We shall show that $x \in U$ by induction on n . Let $n=1$. If $u^{a_1} \in A \setminus H$ then for (2) to hold we must have $b_1 \in H$, for otherwise $l(x^g)=3$. Now $U \triangleleft A$. Therefore $x=u^g \in U$. Hence we need only consider the case when $u^{a_1} \in H$. But this implies that $u^{a_1} \in U \cap H = V \cap H$.

Case 1. $u^{a_1 b_1} \in H$. Since $u^{a_1} \in V$ and $V \triangleleft B$ we have $u^{a_1 b_1} \in V \cap H \subseteq U$. Hence $x=u^g \in U$.

Case 2. $u^{a_1 b_1} \notin H$. Since $u^{a_1} \in H$ we must have $u^{a_1 b_1} \in B \setminus H$. But this contradicts the assumption that $u^g = x \in A$. Hence this case cannot occur.

Let $n > 1$. Let $g_1 = a_1 b_1 \cdots a_{n-1} b_{n-1}$. If $l(u^{g_1}) > 1$, then since $b_{n-1} \in B \setminus H$ and $a_n \in A \setminus H$ it follows that $l(u^g) > 1$ whence (2) cannot hold. Therefore $l(u^{g_1}) \leq 1$. Now $b_{n-1} \in B$ implies that $u^{g_1} \in B$. If $u^{g_1} \in B \setminus H$ then since $a_n \in A \setminus H$ we have $l(u^{g_1}) > 1$. Thus $u^{g_1} \in H$ whence $u^{g_1} \in U^G \cap H \subseteq U^G \cap A$. Hence by induction $u^{g_1} \in U \cap A$. Writing $u^{g_1} = u_1 \in U$ we have $u_1^{a_n b_n} \in U$. Hence $x = u_1^{a_n b_n} \in U$ and $U^G \cap A \subseteq U$.

THEOREM 3.3. *Let $G = (A * B)_H$ be residually finite. If A and B have compatible H -filters and H contains no nontrivial G -normal subgroup, then $\Phi(G) = 1$.*

PROOF. Let $\{U_i; i \in I\}$ and $\{V_i; i \in I\}$ be compatible H -filters of A and B respectively. Let $N_i = U_i^G$ and $G_i = G/N_i$. By Lemma 3.1, $G_i = (A_i * B_i)_{H_i}$ where $A_i = AN_i/N_i$, $B_i = BN_i/N_i$ and $H_i = HN_i/N_i$. It is not difficult to show that the H -filters can be chosen so that each G_i is a nontrivial generalized free product. Now A/U_i is finite, whence H_i is finite. Let M_i be the maximal G_i -normal subgroup contained in H_i . Then by Theorem 1.1, $\Phi(G_i) \subseteq M_i$. If L_i is the set of preimages of M_i in G then $\Phi(G) \subseteq L_i$, whence $\Phi(G) \subseteq L$, where $L = \bigcap_I L_i$.

Now $L_i \subseteq N_i H$. Thus $L_i \cap A \subseteq N_i H \cap A \subseteq (N_i \cap A)H$. Because of the compatibility of the H -filters, $U_i \cap H = V_i \cap H$. Therefore, by Lemma 3.2, $N_i \cap A = U_i$, whence $L_i \cap A \subseteq U_i H$. Since $\{U_i; i \in I\}$ is an H -filter, we have $\bigcap_I U_i H = H$. Hence,

$$L \cap A \subseteq \bigcap_I (L_i \cap A) \subseteq \bigcap_I U_i H = H.$$

Let $x \in \Phi(G) \cap H$ and $a \in A$. Since $L \triangleleft G$, we have $x^a \in L$, whence $x^a \in L \cap A \subseteq H$. But $x^a \in \Phi(G)$. Therefore $x^a \in \Phi(G) \cap H$. It follows that $\Phi(G) \cap H \triangleleft A$. In the same way we can show that $\Phi(G) \cap H \triangleleft B$. Hence $\Phi(G) \cap H \triangleleft G$. But H contains no nontrivial G -normal subgroup. Therefore $\Phi(G) \cap H = 1$, whence by Theorem 2.1, $\Phi(G) = 1$.

COROLLARY 3.4. *Let $G = (A * B)_H$ be residually finite. If A and B have compatible H -filters then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H .*

In [4] Gregorac showed that if $G=(A * B)_H$ is residually finite and A and B each satisfies a suitable identical relation then A and B have compatible H -filters. In fact the suitable identical relation required is a nontrivial identical relation which is not satisfied by the infinite dihedral group. Thus applying Theorem 2 [4] and Corollary 3.4, we have:

THEOREM 3.5. *Let $G=(A * B)_H$ be residually finite. Then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H if one of the following conditions is satisfied:*

- (i) *A and B each satisfies a nontrivial identical relation which is not satisfied by the infinite dihedral group;*
- (ii) *A and B each satisfies a nontrivial identical relation and both $[A:H]$ and $[B:H]$ are greater than 2;*
- (iii) *$[A:H]=[B:H]=2$.*

PROOF. (i) and (ii) are direct consequences of Theorem 2 [4] and Corollary 3.4. In the case of (iii) we note that $H \triangleleft G$. Thus, if $\{S_i; i \in I\}$ is a filter for G , then $U_i = S_i \cap H$ is normal and of finite index in both A and B . Hence $\{U_i; i \in I\}$ is an H -filter for both A and B .

We can apply Theorem 3.5 to various results on residual finiteness of generalized free products. For example, by Theorem 5 of Baumslag [1] and Theorem 6 of Dyer [3], we have the following results:

THEOREM 3.6. *Let $G=(A * B)_H$ where A and B are finitely generated torsion-free nilpotent groups and H is closed in both A and B . Then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H .*

THEOREM 3.7. *Let $G=(A * B)_H$ where $A, B \in \mathcal{F} \cdot \mathcal{A}$ ($\mathcal{F}$ being the class of finite groups and $\mathcal{A}$ the class of finitely generated abelian groups). Then $\Phi(G)$ is contained in the maximal G -normal subgroup contained in H .*

4. In the preceding section results on the Frattini subgroups of residually finite generalized free products are obtained by applying Theorem 3.5. Unfortunately Theorem 3.5 requires A and B each to satisfy a nontrivial identical relation. Thus, in particular, if A and B are free groups we cannot make use of this theorem. In this section we shall derive some results in this direction.

LEMMA 4.1. *Let F be a free group and H be a subgroup of F such that for all $x \in F$ there exists $1 \neq h \in H$ and that $h^x \in H$. Then every nontrivial normal subgroup of F intersects H nontrivially.*

PROOF. Let N be a nontrivial normal subgroup of F . Suppose $N \cap H = 1$. Let $1 \neq x \in N$. Then there exists $1 \neq h \in H$ such that $h^x \in H$. This implies that $[h, x] \in H$. But $N \triangleleft F$. It follows that $[h, x] \in N \cap H = 1$, whence h

and x commute. Now F is free. The centralizer of every nontrivial element of F is cyclic. This implies that h and x belong to the same cyclic group. Thus $\langle h \rangle \cap \langle x \rangle \neq 1$, whence $H \cap N \neq 1$ contradicting the initial assumption.

LEMMA 4.2. *Let F be a free group and H be a finitely generated subgroup of F such that for all $x \in F$ we have $H \cap H^x \neq 1$. Then H is of finite index in F .*

PROOF. $H \cap H^x \neq 1$ for $x \in F$ implies that there exists $1 \neq h \in H$ such that $h^x \in H$. Thus if N is any nontrivial normal subgroup of F then, by Lemma 4.1, $N \cap H \neq 1$. Now H is finitely generated, whence by Theorem 3 [6], H is of finite index in F .

THEOREM 4.3. *Let $G = (A * B)_H$ where A and B are free groups and H is finitely generated such that at least one of $[A:H]$ and $[B:H]$ is infinite. Then $\Phi(G) = 1$.*

PROOF. We shall assume $[A:H]$ to be infinite. Suppose there does not exist $x \in A$ such that $H \cap H^x = 1$. Then, by Lemma 4.2, $[A:H]$ is finite. Hence there exists $x \in A$ such that $H \cap H^x = 1$. Thus by the Proposition 2.3 [8], $\Phi(G) = 1$.

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