A SIMPLE EXAMPLE OF A UNIVERSAL SCHWARTZ SPACE

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ABSTRACT. A Schwartz space E is universal if every Schwartz space is topologically isomorphic to a linear subspace of some power E' of E . In this paper concrete examples of universal Schwartz spaces are exhibited.

In [2, Theorem 8, p. 800] J. Diestel, S. A. Morris and S. A. Saxon announced that there exist universal Schwartz spaces and indicated that a concrete example of such a space had not as yet been exhibited. In this paper we exhibit three such spaces the simplest of which is the following: Take the Banach space c_0 of zero-convergent scalar valued sequences and equip c_0 with the topology $\mathcal{S}$ defined by seminorms of the form

$$q_\lambda(\mu) = \sup |\lambda_n| |\mu_n|$$

where λ ranges over all of the elements of c_0 . The space c_0 equipped with the topology $\mathcal{S}$ is then a universal Schwartz space. (For the definition and basic properties of Schwartz spaces we refer the reader to [3, Chapter 3, §15, pp. 271–288].)

If E is a locally convex space (all of our spaces are assumed to be Hausdorff), then $\tau(E, E')$ denotes the Mackey topology on E with respect to the dual system $\langle E, E' \rangle$.

A seminorm q on a locally convex space E is precompact if there is a sequence λ in c_0 and an equicontinuous sequence $\{a_n\}$ in the topological dual E' of E such that, for each x in E ,

$$q(x) \leq \sup |\lambda_n| |\langle x, a_n \rangle|.$$

(For the basic properties of precompact seminorms we refer the reader to [6].) A linear operator $T: E \rightarrow F$ from one locally convex space into another is precompact [respectively, compact] if T transforms a neighborhood of 0 in E into a precompact [respectively, relatively compact] subset of F . In [6, Corollary 2.10] we showed that a linear operator

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$T: E \rightarrow F$ from one normed space into another is precompact if and only if there is a precompact seminorm q on E such that, for each x in E , $\|Tx\| \leq q(x)$. Terzioglu [8, (4), p. 237] (and, independently, the author [6, 3.1, p. 95]) has shown that: A locally convex space E is a Schwartz space if and only if each continuous seminorm on E is precompact.

Let l_∞ denote the Banach space (sup norm $\|\cdot\|_\infty$) of bounded scalar valued sequences and let $C[0, 1]$ denote the Banach space (sup norm) of continuous scalar valued functions on the unit interval $[0, 1]$.

A projective limit $\text{proj} \lim_{\mu \nu} (E_\nu)$ of Banach spaces E_ν (for the definition of projective limit see [3, p. 155]) is compact if for each μ there is a $\nu > \mu$ such that the linear operator $f_{\mu\nu}: E_\nu \rightarrow E_\mu$ is compact. If each $E_\nu = E$, then we say that $\text{proj} \lim_{\mu \nu} (E_\nu)$ is a projective limit of E -spaces and write $\text{proj} \lim_{\mu \nu} (E)$. In [7, Theorem 2.4] we showed that: If E denotes any one of the spaces c_0 , l_∞ or $C[0, 1]$ then a locally convex space F is a Schwartz space if and only if F is topologically isomorphic to a linear subspace of a compact projective limit of E -spaces.

1. PROPOSITION. *Let E be a locally convex space.*

If $\mathcal{S}$ denotes the topology on E defined by the family of $\tau(E, E')$ -precompact seminorms on E , then $\mathcal{S}$ is the strongest Schwartz topology on E compatible with the dual system $\langle E, E' \rangle$.

PROOF. Clearly $\mathcal{S}$ is compatible with the dual system $\langle E, E' \rangle$.

To show that $E[\mathcal{S}]$ is a Schwartz space, it suffices (see [8, (4), p. 237] or [6, 3.1, p. 95]) to show that if λ is a sequence in c_0 and $\{a_n\}$ is a $\tau(E, E')$ -equicontinuous sequence in E' , then there is a sequence μ in c_0 and an $\mathcal{S}$ -equicontinuous sequence $\{b_n\}$ in E' such that, for each x in E ,

$$(*) \quad \sup |\lambda_n| |\langle x, a_n \rangle| \leq \sup |\mu_n| |\langle x, b_n \rangle|.$$

If λ is a sequence in c_0 and $\{a_n\}$ is a $\tau(E, E')$ -equicontinuous sequence in E' , then choose sequences μ and τ in c_0 so that $\lambda_n = \mu_n \tau_n$ for all n . Let $b_n = \tau_n a_n$. Since

$$\langle x, b_n \rangle \leq \sup |\tau_n| |\langle x, a_n \rangle|$$

it follows that $\{b_n\}$ is an $\mathcal{S}$ -equicontinuous sequence in E' that satisfies (*). Therefore, $E[\mathcal{S}]$ is a Schwartz space and it is easy to see that $\mathcal{S}$ is the strongest Schwartz topology on E compatible with the dual system $\langle E, E' \rangle$.

2. THEOREM. *Let E denote any one of the spaces c_0 , l_∞ or $C[0, 1]$.*

If $\mathcal{S}$ denotes the topology on E defined by the family of precompact seminorms on E , then $E[\mathcal{S}]$ is a universal Schwartz space.

PROOF. Since the norm topology on E coincides with the $\tau(E, E')$ topology on E , Proposition 1 implies that $E[\mathcal{S}]$ is a Schwartz space.

Let F be a Schwartz space. By [7, Theorem 2.4], F is topologically isomorphic to a linear subspace of a compact projective limit $\text{proj} \lim f_{\mu\nu}(E)$ of E -spaces (where μ, ν belong to a given directed set I). Let T denote the natural embedding of $P = \text{proj} \lim f_{\mu\nu}(E)$ into E^I and let S denote the identity operator from E^I into $E[\mathcal{S}]^I$. Since $\mathcal{S}$ is weaker than the norm topology on E , S is continuous. To complete the proof we show that ST is open.

Let μ be an element of I and let U denote the set of all points x in P with $\|x_\mu\| \leq 1$. Choose $\nu > \mu$ in I so that $f_{\mu\nu}: E \rightarrow E$ is compact. Let q be a precompact seminorm on E such that $\|f_{\mu\nu}(x)\| \leq q(x)$ for all x in E . Let V denote the set of all x in $E[\mathcal{S}]^I$ such that $q(x_\mu) \leq 1$, then it is easy to see that

$$V \cap ST(P) \subset ST(U).$$

Therefore, ST is open and F is topologically isomorphic to a linear subspace of $E[\mathcal{S}]^I$.

3. COROLLARY. *If c_0 is equipped with the topology $\mathcal{S}$ defined by seminorms of the form*

$$q_\lambda(\mu) \leq \sup |\lambda_n| |\mu_n|$$

where λ ranges over all of the elements of c_0 , then $c_0[\mathcal{S}]$ is a universal Schwartz space.

PROOF. This follows from Theorem 2 above and the following consequence of [1, Theorem 2.4, p. 477]:

If q is a precompact seminorm on c_0 , then

(*)(*) there is a sequence λ in c_0 such that, for each μ in c_0 ,

$$q(\mu) \leq \sup |\lambda_n| |\mu_n|.$$

4. REMARK. By using (*)(*) above we have been able to show that every precompact linear operator $T: c_0 \rightarrow E$ from c_0 into a normed space E has a representation of the form $T(\mu) = \sum \lambda_n \mu_n y_n$ where λ is a sequence in c_0 and $\{y_n\}$ is a (weakly) unconditionally summable sequence in E . Moreover, c_0 is the only Banach space with basis that has this property. The details will appear elsewhere.

In view of the main result of [6] we pose the following question: If an $\mathcal{L}_\infty$ -space E (see [4] and [5] for the definition and basic properties of $\mathcal{L}_\infty$ -spaces) is equipped with the topology $\mathcal{S}$ defined by the family of precompact seminorms on E , is $E[\mathcal{S}]$ a universal Schwartz space? It has been conjectured (see [5, Problem 3d, p. 346]) that every infinite dimensional $\mathcal{L}_\infty$ -space contains a subspace isomorphic to c_0 . A positive answer to this conjecture would imply a positive answer to our question.

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REFERENCES

1. J. B. Conway, *The strict topology and compactness in the space of measures. II*, Trans. Amer. Math. Soc. **126** (1967), 474–486. MR **34** #6503.
2. J. Diestel, S. A. Morris and S. A. Saxon, *Varieties of locally convex topological vector spaces*, Bull. Amer. Math. Soc. **77** (1971), 799–803.
3. J. Horváth, *Topological vector spaces and distributions*. Vol. I, Addison-Wesley, Reading, Mass., 1966. MR **34** #4863.
4. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$ -spaces and their applications*, Studia Math. **29** (1968), 275–326. MR **37** #6743.
5. J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_p$ -spaces*, Israel J. Math. **7** (1969), 325–349. MR **42** #5012.
6. D. J. Randtke, *Characterizations of precompact maps, Schwartz spaces and nuclear spaces*, Trans. Amer. Math. Soc. **165** (1972), 87–101.
7. ———, *A structure theorem for Schwartz spaces*, Math. Ann. (to appear).
8. T. Terzioglu, *On Schwartz spaces*, Math. Ann. **182** (1969), 236–242. MR **40** #683.

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