MACKEY COMPACTNESS IN BANACH SPACES

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ABSTRACT. *If A' , a subset of a conjugate Banach space X' , is sequentially compact in the Mackey topology $(\tau(X', X))$, then A' is conditionally compact in the Mackey topology. The converse is not true.*

A subset A is conditionally compact if its closure is compact; A is sequentially compact if each sequence of elements of A contains a subsequence converging to an element of the Banach space X . It is well known (Eberlein-Smulian Theorem) that sequential and conditional compactness are equivalent in the weak topology $(\sigma(X, X'))$ of a Banach space X ; and that sequential compactness implies conditional compactness (but not conversely) for the weak-star topology $(\sigma(X', X))$. It is natural to ask what is the relation, if any, between conditionally compact and sequentially compact sets in the Mackey topology $(\tau(X', X))$ of X' ? Recall that the Mackey topology on X' is that topology generated by the polars of all convex, balanced, and weakly compact sets of X (see [4]).

LEMMA 1 (GROTHENDIECK [2, p. 134]). *Let X be a Banach space and A' be a subset of X' . The following conditions are equivalent:*

- (a) *A' is Mackey conditionally compact.*
- (b) *$\lim_n \sup_{A'} |x'(x_n)| = 0$ for every sequence $\{x_n\}$ in X which weakly converges to 0.*

THEOREM 2. *Let $A' \subseteq X'$ be Mackey sequentially compact. Then A' is Mackey conditionally compact.*

PROOF. Let $\{x_n\} \subseteq X$ weakly converge to 0 and $\{x'_n\}$ be a sequence in A' . We show $\lim_n x'_n(x_n) = 0$. Let W be the closed, balanced convex hull of $S = \{x_n, 0\}$. Now W is the closed convex hull of the set $W_0 = \{\alpha s : \alpha \text{ a scalar, } |\alpha| \leq 1, s \in S\}$, and W_0 is the image under the multiplication map of the

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set $\{|\alpha| \leq 1\} \times S$, which is itself compact in the product space (scalars) $\times$ $(X, \text{weak topology})$. Thus, W_0 is weakly compact, so W is weakly compact by the Krein-Smulian Theorem [1, p. 434]. Since A' is Mackey sequentially compact, there exists a subsequence $\{x'_k\}$ of $\{x'_n\}$ which converges uniformly on W . Actually, we can say every subsequence of $\{x'_n(x_n)\}$ has a subsequence converging to 0. Therefore, $\{x'_n(x_n)\}$ converges to 0 and, by Lemma 1, A' is Mackey conditionally compact.

The following is an example of a weak-star compact set which is not weak-star sequentially compact.

EXAMPLE 3. The unit sphere S'' of $l_\infty[0, 2\pi]$ is not weak-star sequentially compact.

PROOF. Define a sequence in S'' by $x''_n(\gamma) = \sin n\gamma$ for all γ in $[0, 2\pi]$ and for $n = 1, 2, 3, \dots$. Suppose $\{x''_n\}$ has a weak-star Cauchy subsequence $\{x''_{n_k}\}$. Then at least $\lim_{n_k} \sin n_k \gamma$ exists for each γ in $[0, 2\pi]$. But this is not possible [3, p. 143].

EXAMPLE 4. If $A' \subseteq X'$ is Mackey conditionally compact, then A' is not necessarily Mackey sequentially compact.

PROOF. Consider the unit sphere S'' of $l_\infty[0, 2\pi]$. By Example 3, S'' is not weak-star sequentially compact; and hence, cannot be Mackey sequentially compact since the Mackey topology is stronger than the weak-star topology [4, p. 248]. Since weak and norm convergence correspond in $l_1[0, 2\pi]$, S'' is Mackey conditionally compact by Lemma 1.

We show in the following that if X is either reflexive or separable, and if $A' \subseteq X'$ is Mackey conditionally compact, then A' is Mackey sequentially compact.

PROPOSITION 5. *Suppose X is reflexive and $A' \subseteq X'$ is Mackey conditionally compact. Then A' is conditionally compact (and hence, Mackey sequentially compact).*

PROOF. Since X is reflexive, the $\tau(X', X)$ -topology is the same as the $\tau(X', X'')$ -topology on X' . But this is the same as the norm topology on X' [4, p. 249].

PROPOSITION 6. *Let X be separable. Then if $A' \subseteq X'$ is Mackey conditionally compact, A' is Mackey sequentially compact.*

PROOF. Consider the set $\text{cl}(A')$ where cl denotes the closure in the Mackey topology. $\text{cl}(A')$ is Mackey compact, the weak-star topology is coarser than the Mackey topology, and both topologies are Hausdorff; hence, they agree on $\text{cl}(A')$. Since the weak-star topology is metrizable on bounded sets [1, Theorem V.5.1], the result follows.

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