AN INEQUALITY FOR THE DISTRIBUTION OF THE BROWNIAN GRADIENT FUNCTION¹

BURGESS DAVIS

ABSTRACT. The Brownian gradient function of a harmonic function u in the unit disc is shown to be distributionally about as large as the classical area function of u . This distribution function inequality strengthens some integral inequalities of Burkholder and Gundy.

If u is a harmonic function on the open unit disc D the classical area and nontangential maximal functions are defined to be, respectively,

$$A_\sigma(u)(x) = \left[u(0)^2 + \int_{\Gamma_\sigma(x)} |\text{grad } u|^2 \right]^{1/2}$$

and

$$N_\sigma(u)(x) = \sup\{|u(z)| : z \in \Gamma_\sigma(x)\},$$

where $\Gamma_\sigma(x)$ is the interior of the smallest convex region containing the disc of radius $\sigma < 1$ and the point $x = e^{i\theta}$. The Brownian analogues of these functions are

$$S(u) = \left[u(0)^2 + \int_0^\tau |\text{grad } u(B(t))|^2 \right]^{1/2}$$

and

$$u^* = \sup_{0 \leq t < \tau} |u(B(t))|,$$

where $B(t)$ is two dimensional Brownian motion started at 0 and τ is the first time $|B(t)| = 1$.

It is known that for a large class of functions Φ , including all powers $\Phi(\lambda) = \lambda^p$, $p > 0$, there are positive constants c , C depending only on σ and Φ but not on u such that $\int_0^{2\pi} \Phi(A_\sigma(u)) dx$, $\int_0^{2\pi} \Phi(N_\sigma(u)) dx$, and $E\Phi(S(u))$ all lie between $cE\Phi(u^*)$ and $CE\Phi(u^*)$, that is, the four functions are Φ -equivalent. See [1] and [2].

The two maximal functions have been shown to be still more strongly related in [1], where it is proved that there are positive constants c' and C'

Received by the editors January 24, 1972 and, in revised form, May 16, 1972.

AMS (MOS) subject classifications (1969). Primary 3111, 6062.

Key words and phrases. Harmonic function, area function, Brownian motion, distribution inequality.

¹ Research under NSF Grant GP-19222.

© American Mathematical Society 1973

depending only on σ such that, for $\lambda > 0$,

$$(1) \quad c'P(u^* > \lambda) \leq m(N_\sigma(u) > \lambda) \leq C'P(u^* > \lambda).$$

Here the following one-sided distribution function inequality for the area function will be proved.

THEOREM 1. *There is a positive constant K depending only on σ such that*

$$(2) \quad m(A_\sigma(u) > K\lambda) \leq KP(S(u) > \lambda)$$

is true if λ is a positive number and u is harmonic on D .

Inequality (2) immediately implies

$$(3) \quad E\Phi(A_\sigma(u)) \leq C_{\sigma, \Phi} E\Phi(S(u))$$

for the class of functions Φ mentioned above.

The converse of (2) is easily seen to be false, for if $u(x) = x$ then $|\text{grad } u| = 1$ so that $A_\sigma(u)^2 \leq \pi$, while $S(u)^2 = \tau$. Thus a converse of (2) would require $P(\tau > C)$ to be 0 for some $C > 0$. However, $P(\tau > C) > 0$ for all $C > 0$.

The right-hand side of (1) and inequality (2) hold for harmonic functions in two variables only. Burkholder and Gundy show, in a paper in preparation, that their analogues in higher dimensions are not true.

The half-plane analogue of (2) is the following: Let P_{x+iy} be the probability measure associated with a Brownian motion $B(t)$ started at $x+iy$, and let η be the first time $B(t)$ is real. For $0 < \sigma$ and x_0 real, define $R_\sigma(x_0)$ to be the cone $\{x+iy: y > 0, |x-x_0| < \sigma y\}$. Then there is a positive constant K' , depending only on σ , such that, for all u harmonic in the upper half-plane and $\lambda > 0$,

$$(4) \quad m\left\{x: \int_{R_\sigma(x)} |\text{grad } u|^2 > K'\lambda\right\} \\ \leq K' \sup_{y \geq 0} \int_{-\infty}^{\infty} P_{x+iy} \left(\int_0^\eta |\text{grad } u(B(t))|^2 > \lambda \right) dx.$$

We omit the proof, since it is a natural extension of the disc result, using the techniques of [1].

I want to thank Richard F. Gundy for many helpful comments.

In the following, $C_1, C_2, \dots$ stand for positive constants depending at most on σ . We use τ to indicate the first time a process hits the unit circle. It is assumed with no essential loss of generality that $u(0) = 0$.

If E_y denotes expectation taken with respect to the probability distribution associated with a Brownian motion $B(t)$ started at $y \in D$ and if f is a

nonnegative continuous function on D , let $\bar{E}_y f = E_y \int_0^\tau f(B(t)) dt$. A formula in Hunt [5, p. 309, equation (2)] states that

$$(5) \quad \bar{E}_y f = \int_D g_D(y, z) f(z) dz,$$

where g_D is the Green's function for D . If S_y is the disc of radius 2 and center y then $S_y \supset D$ so, for each $z \in D$, we have

$$g_D(z, y) \leq g_{S_y}(z, y) = -\log(|z-y|/2) \leq -\log(|z| - |y|/2).$$

Substituting this inequality in (5) we have

$$(6) \quad \bar{E}_y f \leq - \int_0^1 r \log(|r - r_0|/2) dr \int_0^{2\pi} f(re^{i\theta}) d\theta, \quad |y| = r_0 < 1.$$

If $|y| \leq \frac{1}{4}$ and R_y is the disc of radius $\frac{3}{4}$ around y , then $R_y \subset D$, so that, for $z \in R_y$, $g_D(z, y) \geq g_{R_y}(z, y) = -\log(4|y-z|/3)$. In particular if $|z| \leq \frac{1}{4}$ then $|y-z| \leq \frac{1}{2}$ and $g_D(z, y) \geq -\log(\frac{2}{3})$, implying

$$(7) \quad \bar{E}_y f \geq - \int_0^{1/4} \log(\frac{2}{3}) r dr \int_0^{2\pi} f(re^{i\theta}) d\theta, \quad |y| \leq \frac{1}{4}.$$

LEMMA. If ρ is a continuous nonnegative subharmonic function on D such that $\rho(0)=1$ and if $|z| \leq \frac{1}{4}$ then

$$P_z \left(\int_0^\tau \rho(B(t)) dt \geq C_1 \right) \geq C_1.$$

PROOF. Let

$$g(r) = \frac{1}{2\pi} \int_0^{2\pi} \rho(re^{i\theta}) d\theta, \quad 0 < r < 1, \quad g(0) = 1.$$

Since ρ is subharmonic, $g(r) \geq \rho(0)=1$, $0 \leq r < 1$, and g is continuous on $[0, 1)$ by the bounded convergence theorem. Therefore $\hat{\rho}(re^{i\theta}) = \rho(re^{i\theta})/g(r)$ is a nonnegative, continuous function on D such that $\hat{\rho} \leq \rho$ and $\int_0^{2\pi} \hat{\rho}(re^{i\theta}) d\theta = 1$, $0 < r < 1$. We thus have, by (6) and (7),

$$(8) \quad \bar{E}_y \hat{\rho} \leq C_2, \quad y \in D,$$

and

$$(9) \quad \bar{E}_y \hat{\rho} \geq C_3, \quad |y| \leq \frac{1}{4}.$$

If A is a set let $I(A)$ be the indicator function of A and if f is any function defined on A but perhaps not elsewhere let $\int f I(A) = \int_A f$.

Now let $z_0 \leq \frac{1}{4}$. Let η be the first time before τ that $\int_0^\eta \hat{\rho}(B(t)) dt = C_3/2$.

The strong Markov property and (8) imply

$$\begin{aligned} E_{z_0} \int_{\eta}^{\tau} \hat{\rho}(B(t)) dt I(\eta < \tau) &= E_{z_0}[(\bar{E}_{B(\eta)} \hat{\rho}) I(\eta < \tau)] \\ &\leq E_{z_0}(C_2 I(\eta < \tau)) = C_2 P_{z_0}(\eta < \tau). \end{aligned}$$

Thus

$$\begin{aligned} C_3 &\leq \bar{E}_{z_0} \hat{\rho} = E_{z_0} \left(\int_0^{\min(\eta, \tau)} \hat{\rho}(B(t)) dt \right) + E_{z_0} \left(\int_{\eta}^{\tau} \hat{\rho}(B(t)) dt I(\eta < \tau) \right) \\ &\leq C_3/2 + C_2 P_{z_0}(\eta < \tau). \end{aligned}$$

Therefore

$$P_{z_0} \left(\int_0^{\tau} \hat{\rho}(B(t)) dt \geq C_3/2 \right) \geq P_{z_0}(\eta < \tau) \geq C_3/2C_2.$$

Since $\hat{\rho} \leq \rho$, we may take $C_1 = \min(C_3/2, C_3/2C_2)$.

The hypotheses of this lemma were simplified to avoid notation. However the analogue of (5) holds for regions R other than D and in particular for $D(a, r) = \{z: |z-a| < r\}$. Thus a proof exactly like the proof just given can be made to show that if γ is the first time $|B(t)-a|=r$ and if q is a nonnegative subharmonic function on $D(a, r)$ then

$$(10) \quad P_{\gamma} \left(\int_0^{\gamma} q(B(t)) dt \geq C_1 q(a) r^2 \right) \geq C_1, \quad |y-a| \leq r/4.$$

If τ_a is the first time that $|B(t)|=a$ then $B(\tau_a)$ is uniformly distributed on the circle $\{z: |z|=a\}$ under P_0 . Thus, if $D(a, r) \subset D$,

$$\begin{aligned} (11) \quad P_0(|B(t)-a|=r \text{ for some } t, 0 \leq t < \tau) &= P_0(B(t) \in D(a, r) \text{ for some } t, 0 \leq t < \tau) \\ &\geq P_0(B(\tau_a) \in D(a, r)) \geq 2r/2\pi a \geq r/\pi. \end{aligned}$$

Let g stand for $|\text{grad } u|^2$. Let v be the first time $|B(t)-a|=r/4$ and μ the first time after v but before $|B(t)-a|>r$ that $\int_v^{\mu} g(B(t)) dt \geq C_1 r^2 g(a)$. The strong Markov property, (10) and (11) imply

$$\begin{aligned} (12) \quad P_0 \left(\int_0^{\tau} g(B(t)) I(D(a, r)) dt \geq C_1 r^2 g(a) \right) &\geq E_0(P_{B(v)}(\mu < \tau) I(v < \tau)) \\ &\geq E_0(C_1 I(v < \tau)) \\ &= C_1 P_0(v < \tau) \geq C_1 r/\pi. \end{aligned}$$

Let P_0^{θ} be the probability associated with conditional Brownian motion starting at 0 and conditioned by the restriction $B(\tau)=e^{i\theta}$. Conditional

processes of this type have been discussed by Doob in [3] and [4]. The following is a conditional version of Theorem 1 and thus immediately implies it.

THEOREM 2. $P_0^\theta(S^2(u) > C_5 A_\sigma^2(e^{i\theta})) > C_5$.

PROOF. Assume with no loss of generality that $\theta = 0$. Let

$$T_n = \{z \in \Gamma_\sigma(1) : 1 - 2^{-n} < |z| \leq 1 - 2^{-n+1}\}, \quad n \geq 1,$$

and let a_n be a point of the closure of T_n such that $g(a_n) = \sup\{g(z) : z \in T_n\}$. Since for some constant C_6 , and all n , $\text{area } T_n \leq C_6 2^{-2n}$, we have

$$\int_{T_n} g(x) dx \leq C_6 g(a_n) 2^{-2n}$$

so

$$A_\sigma^2(1) = \int_{\bigcup_{n=1}^{\infty} T_n} g(x) dx \leq \sum_{n=1}^{\infty} C_6 g(a_n) 2^{-2n}.$$

Thus either $\sum_{n=1}^{\infty} C_6 g(a_{2n}) 2^{-4n}$ or $\sum_{n=1}^{\infty} C_6 g(a_{2n-1}) 2^{-4n+2}$ is at least half of $A_\sigma^2(1)$. Suppose without loss of generality it is the former. Let μ_{2k} be the first time before τ that

$$\int_0^{\mu_{2k}} g(B(t)) I(D(a_{2k}, 2^{-2k-2})) dt = C_1 g(a_{2k}) 2^{-4k-4}.$$

If $z \in D$ then the density of $B(\tau)$ under P_z on the unit circle is given by the Poisson kernel

$$P(z, e^{i\theta}) = (2\pi)^{-1}((1 - |z|^2)/|z - e^{i\theta}|^2).$$

This is discussed in [3]. If $y \in D(a_{2k}, 2^{-2k-2})$ then, since $a_{2k} \in T_{2k}$, y satisfies $|y| \geq 1 - 2^{-2k+2}$ and $|y - 1| \leq C_7 2^{-2k}$, so the Poisson kernel satisfies

$$(13) \quad P(y, 1) \geq C_8 2^{2k}, \quad y \in D(a_{2k}, 2^{-2k-2}).$$

Let $P_{2k}(0, e^{i\theta})$ be the density of $B(\tau)I(\mu_{2k} < \tau)$ under P_0 . Then since $B(\mu_{2k}) \in D(a_{2k}, 2^{-2k-2})$, (12), (13), and the strong Markov property imply

$$\begin{aligned} P_{2k}(0, 1) &= E_0(P(B(\mu_{2k}), 1)I(\mu_{2k} < \tau)) \\ &\geq C_8 2^{2k} P_0(\mu_{2k} < \tau) \geq C_8 2^{2k} C_1 2^{-2k-2} / \pi = C_9. \end{aligned}$$

Since $P(0, 1) = 1/2\pi$, we have that

$$(14) \quad P_0^0(\mu_{2k} < \tau) = P_{2k}(0, 1)/P(0, 1) \geq C_9/(2\pi)^{-1} = C_{10}.$$

Now if $f = \sum_{k=1}^{\infty} C_1 g(a_{2k}) 2^{-4k-4} I(\mu_{2k} < \tau)$, we have

$$(E_0^0 f^2)^{1/2} \leq \sup |f| \leq \sum_{k=1}^{\infty} C_1 g(a_{2k}) 2^{-4k-4},$$

while

$$E_0^0 f = \sum_{k=1}^{\infty} g(a_{2k}) 2^{-4k-4} P_0^0(\mu_{2k} < \tau) \geq C_{10} \sum_{k=1}^{\infty} g(a_{2k}) 2^{-4k-4}.$$

Thus $(E_0^0 f^2)^{1/2} \leq C_{11} E_0^0 f$, which implies $P_0^0(f \geq C_{12} E_0^0 f) \geq C_{12}$. (See Zygmund, [6, Chapter V, 8.26].)

Since the circles $D(a_{2k}, 2^{-2k-2})$, $k=1, 2, \dots$, are disjoint we have

$$S^2(u) \geq \sum_{k=1}^{\infty} \int g(B(t)) I(D_{a_{2k}}, 2^{-2k-2}) dt \geq \sum_{k=1}^{\infty} C_1 g(a_{2k}) 2^{-4k-4} I(\mu_{2k} < \tau) = f,$$

and since $A_\sigma^2(1) \leq C_{13} E_0^0 f$,

$$P_0^0(S^2(u) \geq C_{12} C_{13} A_\sigma^2(1)) \geq C_{12}.$$

Thus the C_5 in the statement of the theorem may be taken to be $\min(C_{12}, C_{12} C_{13})$. The proof is complete.

REFERENCES

1. D. L. Burkholder, R. F. Gundy and M. L. Silverstein, *A maximal function characterization of the class H^p* , Trans. Amer. Math. Soc. **157** (1971), 137–153. MR **43** #527.
2. D. L. Burkholder and R. F. Gundy, *Distribution function inequalities for the area integral*, Studia Math. (to appear).
3. J. L. Doob, *Semimartingales and subharmonic functions*, Trans. Amer. Math. Soc. **77** (1954), 86–121. MR **16**, 269.
4. ———, *Conditional Brownian motion and the boundary limits of harmonic functions*, Bull. Soc. Math. France **85** (1957), 431–458. MR **22** #844.
5. G. A. Hunt, *Some theorems concerning Brownian motion*, Trans. Amer. Math. Soc. **81** (1956), 294–319. MR **18**, 77.
6. A. Zygmund, *Trigonometrical series*, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

DEPARTMENT OF STATISTICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903