FORMULAS FOR THE NUMBER OF BINOMIAL COEFFICIENTS DIVISIBLE BY A FIXED POWER OF A PRIME

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ABSTRACT. Define $\theta_j(n)$ as the number of binomial coefficients $\binom{n}{s}$ divisible by exactly p^j . A formula for $\theta_2(n)$ is found, for all n , and formulas for $\theta_j(n)$ for $n=ap^k+bp^r$ and $n=c_1p^{k_1}+\cdots+c_mp^{k_m}$ ($k_1\geq j, k_{i+1}-k_i\geq j$ for $i=1, \dots, m-1$) are derived.

1. Introduction. Let p be a fixed prime and let $\theta_j(n)$ denote the number of binomial coefficients $\binom{n}{s}$ ($s=0, 1, \dots, n$) divisible by exactly p^j .

If we put

$$(1.1) \quad n = c_0 + c_1p + \cdots + c_rp^r \quad (0 \leq c_i < p)$$

it is well known [3] that

$$\theta_0(n) = (c_0 + 1)(c_1 + 1) \cdots (c_r + 1).$$

The evaluation of $\theta_j(n)$ for arbitrary j appears to be more difficult, however. Carlitz [1] has proved that

$$\theta_1(n) = \sum_{k=0}^{r-1} (c_0 + 1) \cdots (c_{k-1} + 1)(p - c_k - 1)c_{k+1}(c_{k+2} + 1) \cdots (c_r + 1)$$

and he has found formulas for $\theta_j(n)$ for the following values of n :

$$\begin{aligned} ap^r + bp^{r+1} & \quad (0 \leq a < p, 0 \leq b < p), \\ b + ap + ap^2 + \cdots + ap^{r+j} & \quad (0 < a < p, b = a \text{ or } a - 1). \end{aligned}$$

The writer [4] has considered this problem for $p=2$ and has found formulas for $\theta_j(n)$, $1 \leq j \leq 4$, and for arbitrary j has evaluated $\theta_j(n)$ for a number of special values of n . These formulas are valid only for $p=2$, however.

In this paper we find formulas for $\theta_2(n)$ for all n and for $\theta_j(n)$ for the following values of n :

$$\begin{aligned} ap^k + bp^r & \quad (0 < a < p, 0 < b < p, k < r), \\ c_1p^{k_1} + \cdots + c_mp^{k_m} & \quad (0 < c_i < p, j \leq k_1, j < k_{i+1} - k_i). \end{aligned}$$

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We shall use the following rule, which was proved by Kummer [2, p. 70]. Put

$$(1.2) \quad s = a_0 + a_1p + \cdots + a_rp^r \quad (0 \leq a_i < p),$$

$$(1.3) \quad n - s = b_0 + b_1p + \cdots + b_rp^r \quad (0 \leq b_i < p),$$

$$a_0 + b_0 = c_0 + \varepsilon_0p, \varepsilon_0 + a_1 + b_1 = c_1 + \varepsilon_1p, \cdots,$$

$$\varepsilon_{r-1} + a_r + b_r = c_r + \varepsilon_rp,$$

where each $\varepsilon_i = 0$ or 1. Let N be the exponent of the highest power of p that divides $\binom{n}{s}$. Then we have $N = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_r$.

2. Evaluation of $\theta_2(n)$. If n is given by (1.1) and s and $n-s$ are given by (1.2) and (1.3), it is clear that $N=2$ if and only if exactly two of the ε 's are equal to 1, and $\varepsilon_r=0$. There are two possibilities. Either $\varepsilon_k = \varepsilon_{k+1} = 1$ for some $0 \leq k \leq r-2$ and all other ε 's = 0 or $\varepsilon_k = 1, \varepsilon_m = 1$ for some $0 \leq k \leq r-3, k+2 \leq m \leq r-1$ and all other ε 's = 0. In the first case we can take

$$a_k = c_k + 1, \cdots, p-1; \quad a_{k+1} = c_{k+1}, \cdots, p-1;$$

$$a_{k+2} = 0, \cdots, c_{k+2} - 1.$$

So we have $(p-c_k-1)(p-c_{k+1})c_{k+2}$ choices and the remaining a 's can be chosen in A_k ways, where

$$(2.1) \quad A_k = \left[\prod_{i=0}^r (c_i + 1) \right] / [(c_k + 1)(c_{k+1} + 1)(c_{k+2} + 1)].$$

In the second case, we can take

$$a_k = c_k + 1, \cdots, p-1; \quad a_{k+1} = 0, \cdots, c_{k+1} - 1,$$

$$a_m = c_m + 1, \cdots, p-1; \quad a_{m+1} = 0, \cdots, c_{m+1} - 1,$$

so there are $(p-c_k-1)c_{k+1}(p-c_m-1)c_{m+1}$ choices. The remaining a 's can be selected in $B_{k,m}$ ways, where

$$(2.2) \quad B_{k,m} = \left[\prod_{i=0}^r (c_i + 1) \right] / [(c_k + 1)(c_{k+1} + 1)(c_m + 1)(c_{m+1} + 1)].$$

Thus we have

$$(2.3) \quad \theta_2(n) = \sum_{k=0}^{r-2} (p-c_k-1)(p-c_{k+1})c_{k+2}A_k$$

$$+ \sum_{m=k+2}^{r-1} \sum_{k=0}^{r-3} (p-c_k-1)c_{k+1}(p-c_m-1)c_{m+1}B_{k,m},$$

where A_k and $B_{k,m}$ are defined by (2.1) and (2.2) respectively.

For example,

$$\begin{aligned}\theta_2(a + bp + cp^2) &= (p - a - 1)(p - b)c, \\ \theta_2(a + bp + cp^2 + dp^3) &= (p - a - 1)(p - b)c(d + 1) \\ &\quad + (a + 1)(p - b - 1)(p - c)d \\ &\quad + (p - a - 1)b(p - c - 1)d.\end{aligned}$$

This method does not appear to be very practical for evaluating $\theta_j(n)$ for $j > 2$.

3. Special evaluations. We can use Kummer's theorem to evaluate $\theta_j(ap^k + bp^r)$, where $r > k$, $0 < a < p$, $0 < b < p$. Suppose $k \geq j$ and $r - k > j$. Then there are three ways to have exactly j of the ε 's equal to 1:

- (1) $\varepsilon_{r-j} = \varepsilon_{r-j+1} = \cdots = \varepsilon_{r-1} = 1$, all other ε 's = 0;
- (2) $\varepsilon_{k-j} = \varepsilon_{k-j+1} = \cdots = \varepsilon_{k-1} = 1$, all other ε 's = 0;
- (3) $\varepsilon_{k-m} = \varepsilon_{k-m+1} = \cdots = \varepsilon_{k-1} = 1$, $\varepsilon_{r-h} = \cdots = \varepsilon_{r-1} = 1$, $1 \leq m \leq j-1$, $h = j-m$, all other ε 's = 0.

If s is given by (1.2), in the first case we can take

$$\begin{aligned}a_{r-j} &= 1, \cdots, p-1; & a_i &= 0, \cdots, p-1 & (r-j+1 \leq i \leq r-1); \\ a_r &= 0, \cdots, b-1; & a_k &= 0, \cdots, a,\end{aligned}$$

so there are $(p-1)p^{j-1}(a+1)b$ choices. Using similar reasoning in the other two cases, we have

$$\begin{aligned}\theta_j(ap^k + bp^r) &= (p-1)p^{j-1}(a+1)b \\ (3.1) \quad &+ (p-1)p^{j-1}a(b+1) + (j-1)(p-1)^2p^{j-2}ab \\ &\quad (k \geq j, r > k+j).\end{aligned}$$

Similarly we have

$$(3.2) \quad \begin{aligned}\theta_j(ap^k + bp^r) \\ = (p-1)p^{j-1}(a+1)b + k(p-1)^2p^{j-2}ab \quad (k < j, r > k+j),\end{aligned}$$

$$(3.3) \quad = (p-a-1)p^{j-1}b + k(p-1)^2p^{j-2}ab \quad (k < j, r = k+j),$$

$$(3.4) \quad \begin{aligned}= (p-1)p^{j-1}a(b+1) + (p-1)p^{j-2}(p-a)b \\ + (r-k-1)(p-1)^2p^{j-2}ab \quad (k \geq j, r < k+j),\end{aligned}$$

$$(3.5) \quad \begin{aligned}= (p-1)p^{j-1}a(b+1) + (p-a-1)p^{j-1}b \\ + (j-1)(p-1)^2p^{j-2}ab \quad (k \geq j, r+k=j),\end{aligned}$$

$$(3.6) \quad \begin{aligned}= (p-1)p^{j-2}(p-a)b + (r-j)(p-1)^2p^{j-2}ab \\ (k < j, r < k+j, r \geq j).\end{aligned}$$

We next evaluate $\theta_j(n)$ for

$$(3.7) \quad n = c_1p^{k_1} + c_2p^{k_2} + \cdots + c_mp^{k_m} \quad (k_1 \geq j, k_{i+1} - k_i > j).$$

Using Kummer's theorem, we need to determine the number of ways we can have exactly j of the ε 's equal to 1. Let $1 \leq u \leq m$ and choose u of the c_i 's. Call them $c_{i_1}, \dots, c_{i_u}$. Assign to each c_{i_w} a number t_w , $1 \leq t_w$, such that $t_1 + t_2 + \dots + t_u = j$. This can be done in $\binom{j-1}{u-1}$ ways, since there are $\binom{j-1}{u-1}$ different ways of distributing j nondistinct objects into u distinct cells with no cell left empty. We wish to have $\varepsilon_v = 1$ ($v = i_w - h$, $1 \leq h \leq t_w$, $1 \leq w \leq u$) and all other ε 's equal to 0. If s is given by (1.2) we can take

$$\begin{aligned} a_v &= 1, \dots, p-1 & (v = i_w - t_w, 1 \leq w \leq u), \\ &= 0, \dots, p-1 & (v = i_w - h, 1 \leq h \leq t_w - 1, 1 \leq w \leq u), \\ &= 0, \dots, c_v - 1 & (v = i_w, 1 \leq w \leq u). \end{aligned}$$

Thus for a given u and a given selection $i_1, \dots, i_u$, there are

$$\binom{j-1}{u-1} (p-1)^u p^{j-u} c_{i_1} \cdots c_{i_u} (c_1 + 1) \cdots (c_m + 1) / (c_{i_1} + 1) \cdots (c_{i_u} + 1)$$

different ways to have j of the ε 's equal to 1. Therefore

$$(3.8) \quad \theta_j(n) = \sum_{u=1}^m \binom{j-1}{u-1} (p-1)^u p^{j-u} E_u$$

where n is given by (3.7) and

$$(3.9) \quad E_u = \sum c_{i_1} \cdots c_{i_u} (c_1 + 1) \cdots (c_m + 1) / [(c_{i_1} + 1) \cdots (c_{i_u} + 1)],$$

the sum being over all subsets $\{i_1, \dots, i_u\}$ of $\{1, \dots, m\}$ such that $i_1 < i_2 < \dots < i_u$.

For example, if $n = ap^{k_1} + bp^{k_2} + cp^{k_3}$, $k_1 \geq j$, $k_2 - k_1 > j$, $k_3 - k_2 > j$, then

$$\begin{aligned} \theta_j(n) &= (p-1)p^{j-1}[a(b+1)(c+1) + (a+1)b(c+1) + (a+1)(b+1)c] \\ &\quad + (j-1)(p-1)^2 p^{j-2}[ab(c+1) + a(b+1)c + (a+1)bc] \\ &\quad + \binom{j-1}{2} (p-1)^3 p^{j-3} abc. \end{aligned}$$

If n is given by (3.7) and $c_1 = c_2 = \dots = c_k \equiv a$, then (3.8) becomes

$$(3.10) \quad \theta_j(n) = \sum_{u=1}^m \binom{j-1}{u-1} \binom{m}{u} (p-1)^u p^{j-u} (a+1)^{m-u} a^u.$$

If n is given by (3.7), except that $k_1 = j-1$ and $k_{i+1} - k_i = j$, we have, by an argument similar to the one above,

$$(3.11) \quad \theta_j(n) = p^{j-1} F + \sum_{u=2}^m \binom{j-1}{u-1} (p-1)^u p^{j-u} E_u,$$

where E_u is defined by (3.9) and

$$F = \sum_{i=1}^{m-1} (p - c_i - 1)c_{i+1}(c_1 + 1) \cdots (c_m + 1)/[(c_i + 1)(c_{i+1} + 1)].$$

If $k_1 \geq j$ and $k_{i+1} - k_i = j$, then

$$(3.12) \quad \begin{aligned} \theta_j(n) &= (p-1)p^{j-1}c_1(c_2+1) \cdots (c_m+1) \\ &\quad + p^{j-1}F + \sum_{u=2}^m \binom{j-1}{u-1} (p-1)^u p^{j-u} E_u. \end{aligned}$$

If $k_1 = 0$, $k_{i+1} - k_i > j$, then

$$(3.13) \quad \theta_j(n) = \sum_{u=1}^m \binom{j-1}{u-1} (p-1)^u p^{j-u} G_u,$$

where

$$G_u = \sum c_{i_1} \cdots c_{i_u} (c_1 + 1) \cdots (c_m + 1) / [(c_{i_1} + 1) \cdots (c_{i_u} + 1)],$$

the sum being over all subsets $\{i_1, \dots, i_u\}$ of $\{2, \dots, m\}$ such that $i_1 < i_2 < \dots < i_u$.

If n is given by (3.7) and $c_1 = c_2 = \dots = c_m = a$, then (3.11) becomes

$$(3.14) \quad \begin{aligned} \theta_j(n) &= (m-1)a(p-a-1)p^{j-1}(a+1)^{m-2} \\ &\quad + \sum_{u=2}^m \binom{j-1}{u-1} \binom{m}{u} (p-1)^u p^{j-u} (a+1)^{m-u} a^u; \end{aligned}$$

(3.12) becomes

$$(3.15) \quad \begin{aligned} \theta_j(n) &= (p-1)p^{j-1}a(a+1)^{m-1} \\ &\quad + (m-1)(p-a-1)a(a+1)^{m-2}p^{j-1} \\ &\quad + \sum_{u=2}^m \binom{j-1}{u-1} \binom{m}{u} (p-1)^u p^{j-u} (a+1)^{m-u} a^u; \end{aligned}$$

(3.13) becomes

$$(3.16) \quad \theta_j(n) = \sum_{u=1}^{m-1} \binom{j-1}{u-1} \binom{m-1}{u} (p-1)^u p^{j-u} (a+1)^{m-u} a^u.$$

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