MATRICES WITH PRESCRIBED ENTRIES AND EIGENVALUES. I¹

G. N. DE OLIVEIRA

ABSTRACT. It is proved that there exists a matrix (over an appropriate field) with prescribed eigenvalues and a prescribed non-principal diagonal. It is also proved that there exists an $n \times n$ matrix, over an arbitrary field, with prescribed eigenvalues and with $n-1$ prescribed arbitrary entries.

In an $n \times n$ matrix consider the positions $(i, \sigma(i))$, $i=1, \dots, n$, for some permutation σ . This set of positions is called the σ -diagonal. A σ -diagonal is called principal if and only if σ is the identity permutation.

Many papers on matrices with prescribed characteristic polynomial (or eigenvalues) and prescribed entries have been written ([1], [3], [4], [7], [8], [9]). It is well known that there exists an $n \times n$ matrix, over an arbitrary field, with prescribed principal diagonal and characteristic polynomial $f(\lambda) = \lambda^n - c_1 \lambda^{n-1} + \dots$ if and only if the sum of the prescribed entries equals c_1 [1]. In the present paper we shall extend this result (Theorems 1 and 2 below) for the case in which the prescribed diagonal is not principal. In Theorem 3 we generalize for an arbitrary field a theorem in [3]. Our proof is much simpler than that of [3] and gives an effective method for constructing the matrix.

Let σ be a permutation of $(1, \dots, n)$ different from the identity permutation. We wish to prescribe the entries in the positions $(i, \sigma(i))$, $i=1, \dots, n$, of an $n \times n$ matrix. As in [3] let $\sigma = \sigma_1 \cdots \sigma_s$ be the representation of σ as the product of the disjoint cycles σ_k and n_k be the length of σ_k . A permutation on the rows of a matrix followed by the same permutation on the columns does not alter the eigenvalues (we also wish to prescribe the eigenvalues). Hence, applying an appropriate permutation, we may assume, without loss of generality, that

$$\sigma_1 = (1, \dots, n_1), \sigma_2 = (n_1 + 1, \dots, n_1 + n_2), \dots, \\ \sigma_s = (n_1 + \dots + n_{s-1} + 1, \dots, n)$$

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and that $n_i > 2$ for $i = 1, \dots, \alpha$, $n_i = 2$ for $i = \alpha + 1, \dots, \beta$, $n_i = 1$ for $i = \beta + 1, \dots, s$. Of course some of the numbers α , $\beta - \alpha$, $s - \beta$ may be zero but we assume that α and $\beta - \alpha$ do not vanish simultaneously, since otherwise σ would be the identity permutation.

Now we state our results.

THEOREM 1. *Suppose that at least one of the cycles $\sigma_1, \dots, \sigma_s$ has length > 2 . Let Φ be an arbitrary field and $\lambda_1, \dots, \lambda_n, a_1, \dots, a_n \in \Phi$. There exists an $n \times n$ matrix over Φ with eigenvalues $\lambda_1, \dots, \lambda_n$ and a_i in the position $(i, \sigma(i))$, $i = 1, \dots, n$.*

If there is no cycle of length > 2 , a similar result can be proved but a restriction is necessary.

THEOREM 2. *Suppose $\sigma_1, \dots, \sigma_\beta$ are all of length 2, i.e., $\sigma_i = (2i-1, 2i)$, $i = 1, \dots, \beta$. Let $\lambda_1, \dots, \lambda_n, a_1, \dots, a_n$ be elements of the field Φ . Suppose that there is one value i_0 ($\leq \beta$) of i such that the equation*

$$x^2 - (\Theta + \rho)x + a_{2i_0-1}a_{2i_0} + \Theta\rho = 0,$$

where Θ, ρ are two of the λ_k , has a solution in Φ . Then there exists an $n \times n$ matrix over Φ with eigenvalues $\lambda_1, \dots, \lambda_n$ and a_j in the position $(j, \sigma(j))$, $j = 1, \dots, n$.

REMARK. The restriction imposed in the hypothesis of this theorem cannot, in general, be removed. Example: if $n = 2$ and $x^2 - (\lambda_1 + \lambda_2)x + a_1a_2 + \lambda_1\lambda_2 = 0$ has no solution in Φ , there exists no 2×2 matrix over Φ with λ_1, λ_2 as eigenvalues and a_1, a_2 in the positions $(1, 2)$ and $(2, 1)$ respectively.

In these two theorems we prescribe a nonprincipal diagonal. In the following we prescribe only $n-1$ entries but they need not belong to a diagonal.

THEOREM 3. *Let Φ be an arbitrary field and $\lambda_1, \dots, \lambda_n, a_1, \dots, a_{n-1} \in \Phi$. Let $(i_1, j_1), \dots, (i_{n-1}, j_{n-1})$ be $n-1$ arbitrary but distinct positions in an $n \times n$ matrix. There exists an n -square matrix over Φ with eigenvalues $\lambda_1, \dots, \lambda_n$ and a_t in the position (i_t, j_t) , $t = 1, \dots, n-1$.*

The proofs will be carried out by means of the L -transformation we introduced in [5] for constructing stochastic matrices with prescribed eigenvalues. Since [5] has been published in Portuguese we shall define the L -transformation (indeed only the particular case we need here) before proceeding to the proofs.

We denote by $I_k^{(i,j)}$, $i \neq j$, $i, j \leq k$, the matrix obtained from the $k \times k$ identity matrix by adding its i th row to its j th row. The inverse of $I_k^{(i,j)}$ is obtained from $I_k^{(i,j)}$ by replacing its only off diagonal 1 by -1 . If A is a $k \times k$ matrix,

$$I_k^{(i,j)} A I_k^{(i,j)^{-1}}$$

is easily constructed by adding the i th row of A to its j th row and subtracting its j th column from its i th column. Let Φ_r denote the set of all $r \times r$ matrices over the field Φ . Let C and X be respectively $s \times s$ and $m \times s$ fixed matrices, both over Φ . We define an operator $L_{ijn}^{(X,C)}: \Phi_m \rightarrow \Phi_n$, $n = m + s$, as follows. Let $A \in \Phi_m$ and $B = \begin{bmatrix} A & X \\ 0 & C \end{bmatrix}$, where 0 is the $s \times m$ zero block. We put

$$L_{ijn}^{(X,C)} A = I_n^{(i,j)} B I_n^{(i,j)^{-1}}.$$

The matrix so obtained is called the L -transform (or, more rigorously, the $L_{ijn}^{(X,C)}$ -transform) of A . Clearly the characteristic polynomial of $L_{ijn}^{(X,C)} A$ is the product of the characteristic polynomials of A and C , i.e., the eigenvalues of $L_{ijn}^{(X,C)} A$ are those of A together with those of C .

PROOF OF THEOREM 1. We have

$$\sigma_i = (n_1 + \cdots + n_{i-1} + 1, \cdots, n_1 + \cdots + n_i)$$

and the length n_i of σ_i for $i = 1, \cdots, \alpha$ is > 2 . Let A_i , $i \leq \alpha$, be the $(n_i - 1) \times (n_i - 1)$ matrix having $\lambda_{n_1 + \cdots + n_{i-1} + 1}, \cdots, \lambda_{n_1 + \cdots + n_i - 1}$ down the principal diagonal, $a_{n_1 + \cdots + n_{i-1} + 1}, \cdots, a_{n_1 + \cdots + n_i - 2}$ above the principal diagonal (i.e., in the positions $(1, 2), (2, 3), \cdots, (n_i - 2, n_i - 1)$) and zeros elsewhere. Now set

$$C_i = [\lambda_{n_1 + \cdots + n_i}]$$

and

$$X_i = \begin{bmatrix} \lambda_{n_1 + \cdots + n_{i-1} + 1} - \lambda_{n_1 + \cdots + n_i} - a_{n_1 + \cdots + n_i} \\ 0 \\ \cdot \\ \cdot \\ 0 \\ a_{n_1 + \cdots + n_i - 1} \end{bmatrix},$$

where X_i has $n_i - 1$ (> 1) rows. Let

$$B_i = L_{1, n_i, n_i}^{(X_i, C_i)} A_i, \quad i = 1, \cdots, \alpha.$$

The eigenvalues of B_i are $\lambda_{n_1+\dots+n_{i-1}+1}, \dots, \lambda_{n_1+\dots+n_i}$ and the positions $(1, 2), \dots, (n_i, 1)$ are respectively $a_{n_1+\dots+n_{i-1}+1}, \dots, a_{n_1+\dots+n_i}$. Now put $F = B_1 + \dots + B_\alpha$. It is clear that the eigenvalues of F are $\lambda_1, \dots, \lambda_{n_1+\dots+n_\alpha}$, and that the $(j, \sigma(j))$ entry in F is a_j for $j=1, \dots, n_1+\dots+n_\alpha$.

Now, setting $n_1 + \dots + n_\alpha = p$, we can write

$$\sigma_{\alpha+i} = (p + 2i - 1, p + 2i), \quad i = 1, \dots, \beta - \alpha.$$

Let

$$D_1 = \begin{bmatrix} \lambda_{p+1} & a_{p+1} \\ 0 & \lambda_{p+2} \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \\ a_{p+2} & 0 \end{bmatrix} \quad (p \text{ rows}).$$

It can be easily seen that $F_1 = L_{p, p+2, p+2}^{(Y_1, D_1)} F$ has eigenvalues $\lambda_1, \dots, \lambda_{p+2}$ and that its $(j, \sigma(j))$ entry is a_j for $j=1, \dots, p+2$. Next we take

$$D_2 = \begin{bmatrix} \lambda_{p+3} & a_{p+3} \\ 0 & \lambda_{p+4} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \\ a_{p+4} & 0 \end{bmatrix} \quad (p+2 \text{ rows}).$$

The matrix $F_2 = L_{p+2, p+4, p+4}^{(Y_2, D_2)} F_1$ has eigenvalues $\lambda_1, \dots, \lambda_{p+4}$ and its $(j, \sigma(j))$ entry is a_j for $j=1, \dots, p+4$. Continuing in this way we arrive at a $q \times q$ matrix (where $q = n_1 + \dots + n_\beta$) G which has eigenvalues $\lambda_1, \dots, \lambda_q$ and a_j in the position $(j, \sigma(j))$ for $j=1, \dots, q$.

Now take

$$E_1 = [\lambda_{q+1}], \quad Z_1 = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ a_{q+1} - \lambda_{q+1} \end{bmatrix} \quad (q \text{ rows}).$$

The matrix $G_1 = L_{q, q+1, q+1}^{(Z_1, E_1)} G$ has eigenvalues $\lambda_1, \dots, \lambda_{q+1}$ and a_j in the $(j, \sigma(j))$ position for $j=1, \dots, q+1$ (we recall that the (q, q) position in G is not prescribed).

Next take

$$D_2 = [\lambda_{q+2}], \quad Z_2 = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ a_{q+2} - \lambda_{q+2} \\ 0 \end{bmatrix} \quad (q+1 \text{ rows}).$$

Clearly the matrix $G_2 = L_{q,q+2,q+2}^{(Z_2, D_2)} G_1$ has eigenvalues $\lambda_1, \dots, \lambda_{q+2}$ and a_j in the $(j, \sigma(j))$ position for $j=1, \dots, q+2$.

There is no difficulty in seeing that continuing in this way, after a suitable number of steps, we reach a matrix which satisfies the conditions required in the theorem.

PROOF OF THEOREM 2. We may assume, without loss of generality, that the values i_0, Θ, ρ of the theorem are 1, λ_1, λ_2 respectively. If the equation

$$x^2 - (\lambda_1 + \lambda_2)x + a_1a_2 + \lambda_1\lambda_2 = 0$$

has a root in Φ , we can find in Φ two elements a_{11} and a_{22} such that

$$\begin{bmatrix} a_{11} & a_1 \\ a_2 & a_{22} \end{bmatrix}$$

has eigenvalues λ_1, λ_2 . Operating now with this 2×2 matrix as we did with the matrix F appearing in the proof of the preceding theorem, we see that we can construct a matrix satisfying the requirements of the present theorem.

PROOF OF THEOREM 3. The proof of this theorem will be by induction. For $n=2$, the theorem is obviously true. Assume the theorem is valid for $(n-1) \times (n-1)$ matrices. Following [3], there is an integer $s, 1 \leq s \leq n$, such that either (1) there is exactly one prescribed entry in the s th row or in the s th column, but not both, or (2) there are no prescribed entries in the s th row and s th column. Without loss of generality we may assume that $s=n$.

Case 1. This case is solved as case (i) of Theorem 3 in [3].

Case 2. The last row and column have no prescribed entries. By the induction hypothesis there exists an $(n-1) \times (n-1)$ matrix $T = [t_{ij}]$ with eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ and a_i in the position (i, j) , $i, j = 1, \dots, n-1$.

Take

$$C = [\lambda_n], \quad X = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ t_{i_{n-1}, j_{n-1}} - a_{n-1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (n-1 \text{ rows})$$

where $t_{i_{n-1}, j_{n-1}} - a_{n-1}$ is in the i_{n-1} th row of X . The matrix $L_{i_{n-1}, n, n}^{(X, C)} T$ has eigenvalues $\lambda_1, \dots, \lambda_n$ and has the prescribed elements in the right positions.

REMARK. It is easily seen that in constructing the above matrices with the L -transform we may, in general, prescribe more entries than the ones required in our theorems. For this it is sufficient to give appropriate values to the elements of the matrices X appearing in $L_{ijn}^{(X, C)}$.

In closing we mention some open problems: (1) Are the above results valid if we prescribe the characteristic polynomial (possibly with roots not in Φ) instead of the eigenvalues? (2) What happens if we prescribe not only one diagonal but several, or entries not necessarily on a diagonal? (3) We may prescribe some entries and, instead of the eigenvalues, the singular values for example [6]. (4) We may also consider problems in which symmetry is imposed upon the matrix, etc.

The referee points out that some work related to problem (2) has been done by S. Friedland and appears in the Israel J. Math. **11** (1972), 184-189.

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Current address: R. Vasco da Gama, Bloco B, 4^oD, Coimbra, Portugal