ON SYLOW INTERSECTIONS

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ABSTRACT. Let G be a finite group, and let P be its Sylow p -subgroup. Suppose that $O_p(G)=1$ and $\Omega_1(P)$ is abelian. Then there exists a Sylow p -subgroup P_1 of G such that $P \cap P_1 = 1$.

1. Introduction. It is a long-standing conjecture that if G is a non-abelian simple group and P is its S_p -subgroup then $|G| > |P|^2$. The aim of this note is to prove a partial case of the following stronger conjecture.

Conjecture. Let G be a finite group, and let P be its S_p -subgroup. Suppose that $O_p(G)=1$. Then there exists an S_p -subgroup P_1 of G such that $P \cap P_1 = 1$.

If P is abelian, the conjecture was proved by Brodkey [1]. The author generalized Brodkey's result in [2], proving the conjecture under the assumption that $P \cap P_1 \triangleleft P$ for all S_p -subgroups P_1 of G . In this paper we will prove the following:

THEOREM 1. Let i be a fixed positive integer. Then the conjecture holds, provided that $\Omega_i(P) \cap \Omega_i(P_1) \triangleleft \Omega_i(P)$ for all S_p -subgroups P_1 of G .

Since i is arbitrary, the above-mentioned result of [2] follows immediately from Theorem 1. In addition we get:

THEOREM 2. The conjecture holds if $\Omega_i(P)$ is abelian.

THEOREM 3. Under the assumptions of Theorem 1, $|G| > |P|^2$.

In §2, Ω -functor will be defined. Using this definition, Theorem 4 will be stated, from which Theorem 1 follows immediately. §3 will be devoted to a proof of Theorem 4.

2. Ω -functors. Let G be a finite group, and let p be a prime. An Ω -functor (for p) on G is a mapping W of the set of p -subgroups of G into itself, that satisfies the following conditions: for every p -subgroup P of G ,

- (i) $W(P) \subseteq P$,
- (ii) $W(P) \neq 1$ if $P \neq 1$,
- (iii) $W(P^g) = W(P)^g$ for every $g \in G$, and
- (iv) $W(P_1) \subseteq W(P)$ for every p -subgroup P_1 of G such that $W(P_1) \subseteq P$.

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LEMMA 1. Let W be an Ω -functor on G and let $P, P_1, \dots, P_n$ be p -subgroups of G . Then the following statements hold.

- (a) $P_1 \subseteq P$ implies $W(P_1) \subseteq W(P)$,
- (b) $W(P_1 \cap \dots \cap P_n) \subseteq W(P_1) \cap \dots \cap W(P_n)$,
- (c) $P_1 \cap \dots \cap P_n = 1$ iff $1(P_1) \cap \dots \cap W(P_n) = 1$ iff $W(P_1 \cap \dots \cap P_n) = 1$,
- (d) $W(P) \subseteq P_1 \subseteq P$ implies $W(P) = W(P_1)$.

PROOF. (a) follows from (i) and (iv), (b) follows from (a), and (d) follows from (iv) and (a). As for (c), by (i), $P_1 \cap \dots \cap P_n = 1$ implies $W(P_1) \cap \dots \cap W(P_n) = 1$ which by (b) implies $W(P_1 \cap \dots \cap P_n) = 1$. By (ii), the last equation implies $P_1 \cap \dots \cap P_n = 1$.

In §3 we will prove the following:

THEOREM 4. Let W be an Ω -functor for p on G . Then the conjecture holds, provided that $W(P) \cap W(P_1) \triangleleft W(P)$ for all S_p -subgroups P_1 of G .

It is easy to check that Ω_i , $i=1, \dots$, are Ω -functors on G . Therefore, Theorem 1 is a partial case of Theorem 4.

3. Proof of Theorem 4. We may assume that $|P| > 1$ and since $O_p(G) = 1$, there exists a minimal integer r such that:

$$(1) \quad P_1 \cap P_2 \cap \dots \cap P_r = 1,$$

where P_i are S_p -subgroups of G . We will assume that $r > 2$ and derive a contradiction.

By Lemma 1(c) it follows from (1) that

$$(2) \quad W(P_1) \cap W(P_2) \cap \dots \cap W(P_r) = 1$$

and r is minimal such that (2) is satisfied for some S_p -subgroups P_i of G .

Let $C = W(P_1) \cap W(P_2)$; by our assumptions $W(P_i) \subseteq N = N_G(C)$, $i=1, 2$. Let Q be an S_p -subgroup of N containing $P_3 \cap \dots \cap P_r \cap N$ and let D_i , $i=1, 2$, be S_p -subgroups of N containing $W(P_i)$, $i=1, 2$, respectively. It follows by the definition of W , parts (iii) and (iv), and the Sylow Theorem, that $W(P_i) \subseteq W(D_i)$ and $|W(D_i)| \leq |W(P_i)|$; consequently $W(P_i) = W(D_i)$, $i=1, 2$. Thus $C = W(D_1) \cap W(D_2)$ and there exists an S_p -subgroup R of N such that $C = W(Q) \cap W(R)$. It follows then by (2), the definition of Q and Lemma 1(a), that:

$$\begin{aligned} 1 &= W(Q) \cap W(R) \cap W(P_3) \cap \dots \cap W(P_r) \\ &\supseteq W(P_3 \cap \dots \cap P_r \cap N) \cap W(R) \cap W(P_3) \cap \dots \cap W(P_r) \end{aligned}$$

and by Lemma 1(c), since $R \subseteq N$, we get:

$$(3) \quad R \cap P_3 \cap \dots \cap P_r = 1.$$

Let S be an S_p -subgroup of G containing R . By the minimality of r ,

$$S \cap P_3 \cap \cdots \cap P_r \neq 1$$

and, by Lemma 1(c),

$$W(S) \cap W(P_3) \cap \cdots \cap W(P_r) \neq 1.$$

However, since $W(P_1) \subseteq N$, it follows by the definition of W , parts (iii) and (iv), that $W(R) = W(S)$, and consequently

$$W(R) \cap W(P_3) \cap \cdots \cap W(P_r) \neq 1$$

which, in view of Lemma 1(c), contradicts (3).

REFERENCES

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