A COMPARISON THEOREM FOR EIGENFUNCTIONS

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ABSTRACT. A comparison theorem of Sturm's type is obtained for eigenfunctions of general linear elliptic partial differential operators of second order on bounded domains of n -dimensional Euclidean space. The proof is almost immediate from an earlier identity of the author. The theorem is shown to be stronger than some recent theorems of Kurt Kreith.

Let R be a bounded domain in E^n whose boundary ∂R has a piecewise continuous unit normal N at each point. The linear elliptic differential equations

$$(1) \quad lu = \sum_{i,j=1}^n D_i[a_{ij}(x)D_ju] + 2 \sum_{i=1}^n b_i(x)D_iu + c(x)u = 0,$$

$$(2) \quad Lv = \sum_{i,j=1}^n D_i[A_{ij}(x)D_jv] + 2 \sum_{i=1}^n B_i(x)D_iv + C(x)v = 0$$

are under consideration, where $x = (x_1, \dots, x_n) \in R$, $D_i = \partial/\partial x_i$, $i = 1, \dots, n$, the coefficients in (1) and (2) are real and continuous on $\bar{R}$, and the matrices (a_{ij}) and (A_{ij}) are symmetric and positive semidefinite in R . A comparison theorem will be derived for solutions of (1) and (2) satisfying the following homogeneous boundary conditions on ∂R :

$$(3) \quad h[u] = fu + \sum_{i,j=1}^n a_{ij}N_iD_ju = 0,$$

$$(4) \quad H[v] = Fv + \sum_{i,j=1}^n A_{ij}N_iD_jv = 0,$$

respectively, where f and F are continuous functions on ∂R .

Let m , M be the quadratic functionals defined by

$$m[u] = \int_R \left[\sum_{i,j} a_{ij}D_iuD_ju - 2u \sum_i b_iD_iu - cu^2 \right] dx,$$

$$M[u] = \int_R \left[\sum_{i,j} A_{ij}D_iuD_ju - 2u \sum_i B_iD_iu + (g - C)u^2 \right] dx,$$

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with g chosen exactly as in [4], [5] and [7]. The variation of l , h is defined to be

$$V[u] = m[u] - M[u] + \int_{\partial R} (f - F)u^2 ds.$$

THEOREM 1. *If there exists a nontrivial solution u of (1), (3) such that $V[u] > 0$, then every solution of (2), (4) has a zero at some point in $\bar{R}$.*

PROOF. If $v \neq 0$ in $\bar{R}$, integration of the author's identity [7, p. 326, Equation (9)], or [5, p. 612] over R yields the inequality

$$(5) \quad M[u] + \int_{\partial R} F u^2 ds \geq \int_R \sum_{i,j} D_i((u^2/v)A_{ij}D_j v) dx + \int_{\partial R} F u^2 ds$$

(which in fact is (10) of [7]). However, the right side of (5) is equal to $\int_{\partial R} (u^2/v)H[v] ds$ by Green's formula, which is zero by (4). Also Green's formula gives

$$(6) \quad m[u] + \int_{\partial R} f u^2 ds = - \int_R u l u dx + \int_{\partial R} u h[u] ds = 0$$

and subtraction of (5) from (6) yields the contradiction $V[u] \leq 0$.

The preceding simple proof shows that Kreith's statement [4] "the technique of [5] seems to be limited to the case where $u(x)=0$ on ∂R " is unfounded.

When $\partial R \in C^2$, it is known from the Hopf maximum principle that v and $\partial v / \partial N$ cannot both be zero at any point of ∂R , and hence the conclusion of Theorem 1 can be strengthened to " v has a zero in R ." In the case of the boundary condition $u=0$, Allegretto [1] and the author [8] have derived the strong conclusion for domains with C^1 boundaries, for example.

Kreith's theorem [4, p. 549] replaces the hypothesis $V[u] > 0$ of Theorem 1 by $W[u] > 0$, where

$$W[u] = V[u] + \int_R (2u \nabla u \cdot \mathbf{p} + u^2 \nabla \cdot \mathbf{p}) dx - \int_{\partial R} \mathbf{p} \cdot \mathbf{N} u^2 ds$$

for some C^1 vector function $\mathbf{p}$ on $\bar{R}$. Actually, $W[u] = V[u]$ for an arbitrary C^1 vector function $\mathbf{p}$ by Green's formula, so that Kreith's hypothesis about the existence of $\mathbf{p}$ is redundant. Various corollaries of Theorem 1 (which are weaker than Theorem 1) given by Kreith follow by specializing $\mathbf{p}$ so that a certain quadratic form is positive semidefinite in R [4, p. 548].

In the special case that (3) is replaced by the boundary condition $u(x)=0$ on ∂R , Theorem 1 reduces to the author's earlier comparison theorem [5], also stated as Corollary 1 of [4]. Another comparison theorem of Kreith ([3] and [4, Corollary 2]) is the special case that the derivative terms in $V[u]$ are replaced by a lower bound $-g_2(x)u^2(x)$ in R .

If b and B happen to be differentiable functions, integration by parts transforms $V[u]$ to the form

$$V[u] = \int_R \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u + (C - c - g + \nabla \cdot (b - B)) u^2 \right] dx$$

yielding various immediate corollaries of Theorem 1 ([4], [5], [7]).

Theorem 1 can easily be extended to unbounded domains R by the technique of [7], to quasilinear differential equations as in [8], and to differential inequalities as in [7]. We remark that Theorem 1 in the self-adjoint case was obtained in 1966 by the author [6] along with an extension to elliptic operators of even order.

A generalization of some recent theorems of Dunninger and Weinacht [2] also follows immediately from Theorem 1, i.e. from the author's identity ([5, p. 612] and [7, p. 327]) in the special case that every entry of the matrices (a_{ij}) and (A_{ij}) vanishes on a subdomain S' of ∂R . Then obviously (5) and (6) still hold if the boundary conditions (3) and (4) hold only on $\partial R - S'$ (with $f = F = 0$ on S').

COROLLARY. *Let $S' \subset \partial R$ be the "singular part" of ∂R , i.e. $a_{ij} = A_{ij} = 0$ on S' for $i, j = 1, 2, \dots, n$. Then Theorem 1 remains true if the boundary conditions (3) and (4) hold only on $\partial R - S'$.*

As in [7], the same conclusion holds if (1), (2), (3), and (4) are replaced by the inequalities $ulu \geq 0$, $vLv \leq 0$, $uh[u] \leq 0$, and $vH[v] \geq 0$, respectively, since (5) remains true and (6) is replaced by $m[u] + \int_{\partial R} fu^2 ds \leq 0$. The main Theorem 3.1 of Dunninger and Weinacht [2] is the following special case of the corollary:

$$\begin{aligned} b_i &= B_i = 0, \quad i = 1, \dots, n; \quad g = 0, \\ a_{ij} &= y^k \tilde{a}_{ij}, \quad A_{ij} = y^k \tilde{A}_{ij} \quad \text{if } i, j = 1, \dots, n-1 = m; \\ a_{in} &= A_{in} = 0 \quad \text{if } i = 1, \dots, n-1; \\ a_{nn} &= A_{nn} = y^k; \quad c = y^k \tilde{c}, \quad C = y^k \tilde{C}; \\ f &= y^k \tilde{f}, \quad F = y^k \tilde{F}; \quad \text{where } y = x_n. \end{aligned}$$

In this case, the variation $V[u]$ reduces to

$$\begin{aligned} V[u] &= \int_R y^k \left[\sum_{i,j=1}^{n-1} (\tilde{a}_{ij} - \tilde{A}_{ij}) D_i u D_j u + (\tilde{C} - \tilde{c}) u^2 \right] dx \\ &\quad + \int_{S_2} y^k (\tilde{f} - \tilde{F}) u^2 ds, \end{aligned}$$

where S_2 is the component of ∂R on which f and F in the boundary conditions (3) and (4) are finite and nonzero. (The boundary condition $u=0$

corresponds to $f = \infty$ by the usual convention.) Other theorems of Dunninger and Weinacht are special cases or easy consequences of the above results. (On [2, p. 55], the author's method ([5], [7]) of using a quadratic form in $n+1$ variables is mentioned without reference.)

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