ABSOLUTELY CONTINUOUS COMPONENT OF A CLASS OF INTEGRAL OPERATORS¹

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ABSTRACT. The operator $T: L^2(0, \infty) \rightarrow L^2(0, \infty)$ defined by

$$Tf(x) = \int_0^\infty \frac{k(x)(k(t))^-}{x+t} f(t) dt,$$

where $(k(t))^-$ is the complex conjugate, is studied and conditions are given which are sufficient to characterize the absolutely continuous component.

W. Koppleman [4], J. D. Pincus [5], [6], and M. Rosenblum [9] have studied the selfadjoint, singular-integral operator

$$Sf(x) = a(x)f(x) + (\pi i)^{-1} \text{P.V.} \int_E \frac{b(x)(b(t))^-}{x-t} f(t) dt,$$

where E is a Lebesgue measurable subset of $(-\infty, \infty)$ of positive measure, $a(x)$ is bounded and real valued, and $b(x) \in L^\infty(E)$. A complete spectral theory is known for such operators. In particular, the spectral measure is absolutely continuous, except for the possibility of one or two point masses in case $E = (-\infty, \infty)$. Note that formally S has a one-dimensional commutator with the selfadjoint operator X defined by $Xf(x) = xf(x)$. For two-dimensional commutator systems there is no closed form expression for the spectral invariants (see, however, Pincus [7]). In this paper we study a class of integral operators which formally have a two-dimensional commutator with X^2 , and for which we can calculate the absolutely continuous component.

Define $T: L^2(0, \infty) \rightarrow L^2(0, \infty)$ by

$$Tf(x) = \int_0^\infty \frac{k(x)(k(t))^-}{x+t} f(t) dt,$$

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where k is essentially bounded on $(0, \infty)$. J. S. Howland [2] has given necessary and sufficient conditions that T be of trace class. We now characterize the absolutely continuous component of T .

THEOREM. (1) If $x^{-1/2-\delta}(1-k(x)) \in L^2(0, 1)$ and if $x^{-1/2+\delta}(1-k(x)) \in L^2(1, \infty)$ for some $\delta > 0$, then the absolutely continuous component of T has uniform multiplicity 2 on $[0, \pi]$.

(2) If $x^{-1/2-\delta}(1-k(x)) \in L^2(0, 1)$ and if $x^{-1/2+\delta}k(x) \in L^2(1, \infty)$ for some $\delta > 0$, then the absolutely continuous component of T has uniform multiplicity 1 on $[0, \pi]$.

The technique of proof is to find another operator, T_1 , for which the spectral invariants are known and then find conditions on k so that $T_1 - T$ is of trace class. The Rosenblum-Kato theorem (see Kato [3]) then guarantees that the absolutely continuous components of T_1 and T are unitarily equivalent. Consider then the operator $T_1: L^2(0, \infty) \rightarrow L^2(0, \infty)$ defined by

$$T_1 f(x) = \int_0^\infty (x+t)^{-1} f(t) dt.$$

Let L_v denote the multiplication operator on $L^2(-\infty, \infty)$ defined by $L_v f(x) = v(x)f(x)$.

LEMMA 1. T_1 is unitarily equivalent to L_v , where $v(x) = \pi \operatorname{sech}(\pi x)$. Thus, T_1 is absolutely continuous and has uniform multiplicity 2 on $[0, \pi]$.

PROOF. Let $U: L^2(0, \infty) \rightarrow L^2(-\infty, \infty)$ be defined by $Uf(x) = e^{-x/2}f(e^{-x})$. Then U is unitary, and $U^{-1}f(x) = x^{-1/2}f(-\log x)$. Furthermore,

$$UT_1U^{-1}f(x) = e^{-x/2} \int_0^\infty (e^{-x} + t)^{-1} t^{-1/2} f(-\log t) dt.$$

By letting $u = -\log t$, we obtain

$$UT_1U^{-1}f(x) = \int_{-\infty}^\infty \frac{1}{2} \operatorname{sech}\left(\frac{u-x}{2}\right) f(u) du.$$

This convolution is unitarily equivalent, via the Fourier transform, to the desired multiplication operator since the Fourier transform of $\frac{1}{2} \operatorname{sech}(x/2)$ is $(\pi/2)^{1/2} \operatorname{sech}(\pi x)$ (Erdélyi [1]).

Consider now the operator $T_2: L^2(0, \infty) \rightarrow L^2(0, \infty)$ defined by

$$T_2 f(x) = \int_0^\infty e^{-(x+t)/2} (x+t)^{-1} f(t) dt.$$

T_2 is unitarily equivalent to the Hilbert matrix $[1/(i+j+1)]_{i,j=0,1,2,\dots}$, and

Rosenblum [8] proves that this operator has uniform multiplicity 1 on $[0, \pi]$.

LEMMA 2. Let $\phi(x)$ be essentially bounded on $[0, \infty)$ and define the operator $R: L^2(0, \infty) \rightarrow L^2(0, \infty)$ by $Rf(x) = \int_0^\infty e^{-xt} \phi(t) f(t) dt$. If $x^{-1/2-\delta} \phi(x) \in L^2(0, 1)$ and $x^{-1/2+\delta} \phi(x) \in L^2(1, \infty)$, for some $\delta > 0$, then R is of trace class.

PROOF. The Mellin transform can be used to show that R is a bounded operator.

Let P_0 be the orthogonal projection mapping $L^2(0, \infty)$ onto $L^2(0, 1)$, and P_1 be the orthogonal projection mapping $L^2(0, \infty)$ onto $L^2(1, \infty)$. Then

$$Rf(x) = RP_0f(x) + RP_1f(x) = \int_0^1 e^{-xt} \phi(t) f(t) dt + \int_1^\infty e^{-xt} \phi(t) f(t) dt.$$

Note that $RP_0 = B_+ B_-$, where $B_-: L^2(0, 1) \rightarrow L^2(0, 1)$ and $B_+: L^2(0, 1) \rightarrow L^2(0, \infty)$ are defined by

$$B_-f(x) = \int_x^1 t^{-1-\delta} \phi(t) f(t) dt, \quad \text{and}$$

$$B_+f(x) = \int_0^1 \frac{d}{dt} (t^{1+\delta} e^{-xt}) f(t) dt.$$

It is easy to see that B_+ and B_- have square summable kernels, hence they are Hilbert-Schmidt operators and RP_0 is of trace class. Similarly, $RP_1 = K_+ + K_- + K_0$, where $K_-: L^2(1, \infty) \rightarrow L^2(1, \infty)$, $K_+: L^2(1, \infty) \rightarrow L^2(0, \infty)$ are defined by

$$K_-f(x) = \int_x^\infty t^{-1+\delta} \phi(t) f(t) dt,$$

$$K_+f(x) = \int_1^\infty \frac{d}{dt} (t^{1-\delta} e^{-xt}) f(t) dt,$$

and K_0 is the rank one operator defined by

$$K_0f(x) = e^{-x} \int_1^\infty t^{-1+\delta} \phi(t) f(t) dt.$$

Again it is easy to see that K_+ and K_- are Hilbert-Schmidt operators, so RP_1 is of trace class. Hence R is of trace class.

PROOF OF THE THEOREM. To prove part (1) write

$$T_1f(x) = \int_0^\infty (x+t)^{-1} f(t) dt = \int_0^\infty e^{-xu} \int_0^\infty e^{-tu} f(t) dt du = J_1^2,$$

where $J_1: L^2(0, \infty) \rightarrow L^2(0, \infty)$ is defined by $J_1f(x) = \int_0^\infty e^{-xt} f(t) dt$. Now

write $T=J^*J$, where $J:L^2(0, \infty)\rightarrow L^2(0, \infty)$ is defined by $Jf(x)=\int_0^\infty e^{-xt}(k(t))^{-1}f(t) dt$. Then $T_1-T=J_1^2-J^*J=J_1(J_1-J)+(J_1-J^*)J$. But J_1-J is of trace class by Lemma 2, so part (1) follows from the Rosenblum-Kato theorem and Lemma 1.

Similarly, for part (2) write $T_2=J_2^*J_2$, where $J_2:L^2(0, \infty)\rightarrow L^2(0, \infty)$ is defined by $J_2f(x)=\int_0^\infty e^{-t(1+2x)/2}f(t) dt$. Then

$$T_2 - T = J_2^*J_2 - J^*J = J_2^*(J_2 - J) + (J_2^* - J^*)J.$$

We use Lemma 2 to show that J_2-J is of trace class by setting $\phi(x)=e^{-x/2}-k(x)$, and noting that the condition $x^{-1/2-\delta}(1-k(x))\in L^2(0, 1)$ is equivalent to $x^{-1/2-\delta}(e^{-x/2}-k(x))\in L^2(0, 1)$ and $x^{-1/2+\delta}k(x)\in L^2(1, \infty)$ is equivalent to $x^{-1/2+\delta}(e^{-x/2}-k(x))\in L^2(1, \infty)$. Part (2) then follows from the Rosenblum-Kato theorem and the result stated above concerning the Hilbert matrix.

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