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SHORTER NOTES

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A SHORT PROOF OF MAC LANE'S PLANARITY THEOREM

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ABSTRACT. This note gives a short and elementary proof of Mac Lane's theorem on the embedding of graphs in a 2-sphere.

The purpose of this note is to give a short and elementary proof of a theorem by Saunders Mac Lane on the embedding of graphs in the 2-sphere [4]. Existing proofs are the original ones of Mac Lane and an algebraic topology proof by Lefschetz [3]. Our proof is by Kuratowski's theorem [2].

Terminology follows [4] and [1], with the exception that we shall call Mac Lane's 2-fold complete set of circuits a P-base.

Let G be a nonseparable graph.

THEOREM. *If G has a P-base, then G is planar.*

PROOF. Let $C_1, \cdots, C_n$ form a P-base for G, and suppose that G is nonplanar. Then $n>1$ and, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 or to $K_{3,3}$. We claim that H also has a P-base. This is immediate by induction if it is first shown that $G-e$ has a P-base for each arc e of G. But, if e is in exactly one C_i, say C_1, then $C_2, \cdots, C_n$ form a P-base for $G-e$, and if e is in two C_i's, say C_1 and C_2, then $C_3, \cdots, C_n, C_{n+1}=\sum_{i=1}^{n} C_i$ form a P-base for $G-e$.

Thus H, hence also K_5 or $K_{3,3}$, has a P-base. We now show that this is impossible.

If $C_1, \cdots, C_6$ form a P-base for K_5, then each of the ten branches of K_5 is in exactly two of the circuits $C_1, \cdots, C_6, C_7=\sum_{i=1}^{6} C_i$. But each circuit has at least three branches, so

$$\sum_{i=1}^{7} (\text{number of branches in } C_i) = 20 \geqq 21.$$

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Similarly, if $C_1, \cdots, C_4$ form a P-base for $K_{3,3}$, then set $C_5 = \sum_{i=1}^{4} C_i$. Since each circuit in $K_{3,3}$ has at least four arcs, then

$$\sum_{i=1}^{5} (\text{number of branches in } C_i) = 18 \geqq 20,$$

completing the proof.

The converse of the theorem is of course also true, but the proof of this is trivial.

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