AN EXAMPLE OF A HILBERT RING WITH MAXIMAL IDEALS OF DIFFERENT HEIGHT

LESLIE G. ROBERTS

ABSTRACT. An example is given of an integral domain which is a Hilbert ring of Krull dimension two but has some maximal ideals of height one.

1. Introduction. Examples of integral domains which are Hilbert rings with maximal ideals of different height do not seem to be well known. In [3, §5, Example 2], Krull has an example of a Hilbert ring of dimension 2 with one maximal ideal of height 1 that is obtained by localization from $K[X_1, X_2]$, where K is a countable field. In this paper, by localization from $C[X_1, X_2]$, I obtain a Hilbert ring of dimension 2 with an uncountable number of maximal ideals of height 1. By further localization one can obtain a Hilbert ring of dimension 2 with any prescribed finite (or countable) number of maximal ideals of height 1.

By Hilbert ring I mean a ring in which every prime ideal is the intersection of maximal ideals. The height of a prime ideal P is the length n of the longest chain $0 \subseteq P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n = P$ of prime ideals from 0 to P . Note that if the ring is not required to be an integral domain, then one can take a reducible variety with components of different dimensions. R and C denote the real and complex numbers, respectively.

I learned of the problem from [1, p. 4].

2. The example. Let $A = C[Z, W]$, where Z and W are indeterminants. Identify $C^2 = \{(z, w) \mid z, w \in C\}$ with $R^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in R\}$ by $z = x_0 + ix_1$, $w = x_2 + ix_3$. Let $U = \{(z, w) \mid |z|^2 + |w|^2 < 1\}$. This is just the interior of the unit ball in R^4 . Let S be the multiplicative set in A generated by all linear polynomials $aZ + bW + c$ whose variety does not meet U . Then, I claim that A_S is a Hilbert ring of dimension 2 that has an infinite number of maximal ideals of height 1.

$\text{Spec } A_S$ is the subset of $\text{Spec } A$ consisting of all prime ideals that do not meet S . First of all, every closed point P in $\text{Spec } A$ lying outside of U lies on the variety of some element of S . For, let $P = (z_0, w_0)$ with $|z_0|^2 + |w_0|^2 \geq 1$. Then the complex line $\bar{z}_0 Z + \bar{w}_0 W = |z_0|^2 + |w_0|^2$ contains P and

Received by the editors April 12, 1972.

AMS (MOS) subject classifications (1970). Primary 13A15.

© American Mathematical Society 1973

does not meet U , for its real part is just the hyperplane in R^4 through P perpendicular to the line from the origin to P . Thus all closed points in $\text{Spec } A$ outside U have been eliminated so the maximal ideals in A_S of height 2 correspond to the points of U . On the other hand, there exist curves in C^2 which are not a product of complex lines and do not meet U , for any curve not passing through the origin is a positive distance away from the origin, and multiplying all coordinates by a sufficiently large real constant gives a curve missing U . These correspond to maximal ideals of height 1 in A_S . Finally, A_S is a Hilbert ring. To show this, one must prove that every prime of height 1 that is not maximal is contained in infinitely many maximal ideals (by Theorem 147 of [2]). But such ideals correspond to curves in C^2 that meet U . Any such curve meets U in an infinite number of points because U is open, and hence the corresponding ideal is contained in an infinite number of maximal ideals.

In A_S all the maximal ideals of height 1 are principal. Let S' be a set of generators of all but a finite (or countable) number of these ideals. We may choose S' so that $S' \subset A$. Let $T = S \cup S'$. Then A_T will be a Hilbert ring of dimension 2 with a prescribed finite (or countable) number of maximal ideals of height 1.

Finally $A_S[X_1, \dots, X_n]$ (respectively $A_T[X_1, \dots, X_n]$) is a Hilbert ring of dimension $n+2$ with an uncountable (respectively countable or finite) number of maximal ideals of height $n+1$. I do not know if a Krull ring of dimension n can have maximal ideals of height less than $n-1$.

REFERENCES

1. A. V. Geramita, *Maximal ideals in polynomial rings over Dedekind domains*, Queen's University Mathematical Preprint No. 1971-20.
2. I. Kaplansky, *Commutative rings*, Allyn & Bacon, Boston, Mass., 1970. MR 40 #7234.
3. W. Krull, *Jacobsonsche Ringe, Hilbertscher Nullstellensatz, Dimensiontheorie*, Math. Z. 54 (1951), 354-387. MR 13, 903.

DEPARTMENT OF MATHEMATICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA