THE RANGE OF INVARIANT MEANS ON LOCALLY COMPACT GROUPS AND SEMIGROUPS

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ABSTRACT. This paper extends the results of Granirer and Chou concerning the range of a left invariant mean on a discrete semigroup to the case when S is any Borel subsemigroup of a locally compact group.

0. Introduction. Granirer has shown in [2] that for an infinite, right cancellation, left amenable discrete semigroup S , other than what he calls an " AB group", there exists a nested family of left almost convergent subsets of S on which any left invariant mean (LIM) attains all values of the closed interval $[0, 1]$. That is, there exists a family $\{A(t) | t \in [0, 1]\}$ of subsets of S for which

- (i) $s < t$ implies $A(s) \subseteq A(t)$, and
- (ii) $\varphi(\chi_{A(t)}) = t$ for any LIM φ on $m(S)$.

Chou, in [1], partially extended this result to the case when S is a group and obtained the following theorem thereby proving a conjecture made in [2].

THEOREM (CHOU). *If S is an infinite right cancellation left amenable semigroup then the range of each LIM on $m(S)$ is the whole $[0, 1]$ interval.*

In this paper we extend these results to locally compact topological groups and obtain the following main theorems:

THEOREM A. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group. If φ is a LIM on $L^\infty(S)$ then there exists a nested family $\{A(t) | t \in [0, 1]\}$ of Borel subsets of S such that $\varphi(\chi_{A(t)}) = t$ for all $t \in [0, 1]$.*

THEOREM B. *Let S be an infinite Borel subsemigroup in a locally compact group and let A denote the algebra of bounded Borel measurable*

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functions on S . If φ is a LIM on A then there exists a nested family $\{A(t) | t \in [0, 1]\}$ of Borel subsets of S such that $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

These theorems improve Chou's results even for the case that G is discrete, by providing a nested family of subsets on which a given LIM attains all values in the $[0, 1]$ interval. They also supply answers concerning the range of a LIM on such frequently encountered spaces as $L^\infty(R)$ (R the additive reals with the usual topology) to which Granirer's and Chou's results do not apply.

However, the nested sets obtained in these theorems depend heavily on the mean φ being considered and hence are not left almost convergent as are those obtained by Granirer for discrete semigroups.

We have been able to show that if G is abelian and compactly generated, there exists a nested family of left almost convergent Borel sets with the desired property (manuscript in preparation) but we do not know how to obtain such a family for arbitrary locally compact groups.

The results in the body of this paper are somewhat more general than the ones quoted in this introduction.

1. Preliminaries. By a function algebra on a nonvoid set S we mean a sup norm closed, point separating algebra of bounded real-valued functions on S containing the constants. If $(X, \mathcal{S}, \mu)$ is a measure space we denote by $L^\infty(X)$ the algebra of bounded, real-valued, $\mathcal{S}$ -measurable functions on X with the essential sup norm (so $L^\infty(X)$ is not strictly speaking a function algebra).

If S is a semigroup we say that a function algebra A on S is left invariant if $L_t f \in A$ for all $t \in S, f \in A$ where $L_t f(s) = f(ts)$ for all $s \in S$. An element φ in A^* is called a mean if $\|\varphi\| = 1$ and $\varphi(f) \geq 0$ whenever $f \geq 0$ in A . φ is said to be a left invariant mean (LIM) on A if $\varphi(L_t f) = \varphi(f)$ for all $t \in S$ and $f \in A$. If $B \subseteq S$ is such that $\chi_B \in A$ (where χ_B denotes the characteristic function of the set B) then, for convenience of notation, we will often refer to $\varphi(B)$ rather than $\varphi(\chi_B)$ if no confusion will result. A function $f \in A$ is called left almost convergent if $\varphi(f)$ is the same value for any LIM φ on A .

If A is a function algebra (or $L^\infty(X)$ for a measure space $(X, \mathcal{S}, \mu)$) then the structure space $\Delta(A)$ of A is the set of all multiplicative means on A (i.e. those means φ for which $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$ for all $f, g \in A$) and is equipped with the w^* topology as a subset of A^* . Under this topology $\Delta(A)$ is a compact Hausdorff space and A is homeomorphic to the continuous functions on $\Delta(A)$ under the mapping

$$f \rightarrow \hat{f} \text{ where } \hat{f}(\lambda) = \lambda(f) \text{ for all } \lambda \in \Delta(A)$$

(see [5, p. 479] for details).

If S is a topological space we will denote by $B(S)$ the Borel subsets of S (i.e. the smallest σ -field of subsets containing the open sets). A subsemigroup T of S will be called a Borel subsemigroup if $T \in B(S)$.

If $\mathcal{B}$ is a σ -field of subsets of a semigroup S then $\mathcal{B}$ is called left invariant if $t^{-1}B \in \mathcal{B}$ for all $t \in S$ and $B \in \mathcal{B}$ (where $t^{-1}B = \{s \in S | ts \in B\}$). Note that if $\mathcal{B}$ is a left invariant σ -field and A is the algebra of bounded $\mathcal{B}$ -measurable functions then A is a left invariant function algebra.

If S is a topological semigroup, the algebra of left uniformly continuous, bounded functions on S , denoted by $LUC(S)$ consists of all bounded continuous functions f on S with the property that

$$\lim \|L_{s_\alpha}f - L_{s_0}f\| = 0 \quad \text{whenever } s_\alpha \rightarrow s_0 \text{ in } S.$$

Let $\mathcal{B}$ be a σ -field of subsets of a nonvoid set S . If A is the algebra of bounded $\mathcal{B}$ -measurable functions on S then the topology on $\Delta(A)$ can be characterized as follows:

For $B \in \mathcal{B}$ let $U_B = \{\psi \in \Delta(A) | \psi(B) = 1\}$. Then U_B is open-closed in $\Delta(A)$ and $\{U_B | B \in \mathcal{B}\}$ is a base for the topology on $\Delta(A)$. (This can be easily shown using the facts that the simple functions on $\mathcal{B}$ are dense in A and that each $\psi \in \Delta(A)$ is multiplicative which implies that $\psi(B) = 0$ or 1 for all $B \in \mathcal{B}$.) In a similar fashion it can be shown that, for a measure space $(X, \mathcal{S}, \mu)$, $\{U_B | B \in \mathcal{S}\}$ is a base of open-closed sets for the topology on $\Delta(L^\infty(X))$.

2. Probability measures on $\Delta(A)$. If φ is a mean on a function algebra A then, since A is isometrically isomorphic to the continuous functions on the compact, Hausdorff space $\Delta(A)$, there exists a regular probability measure μ_φ on $\Delta(A)$ such that

$$\int_{\Delta(A)} f d\mu_\varphi = \varphi(f) \quad \text{for all } f \in A.$$

Note that when A is the algebra of bounded functions measurable with respect to a σ -field $\mathcal{B}$ then we have $\mu_\varphi(U_B) = \varphi(B)$ for all $B \in \mathcal{B}$.

Let X be locally compact and Hausdorff and μ a regular probability measure (as in [5]) on $B(X)$. If μ is continuous, i.e. $\mu(\{x\}) = 0$ for all $x \in X$, then by [5, p. 132], for all $B \in B(X)$ we have $\{\mu(B') | B' \in B(X), B' \subset B\} = [0, \mu(B)]$. Combining an idea of Chou in [1] and an idea of Granirer in [2] we now prove the following lemma which will be the key tool used in obtaining the main results of this paper.

LEMMA 2.1A. *Let S be a nonempty set. $\mathcal{B}$ a σ -field of subsets of S and A the algebra of bounded, $\mathcal{B}$ -measurable functions on S . Let φ be a mean on A for which the corresponding probability measure μ_φ is continuous.*

Then there exists a family $\{A(t)|t \in [0, 1]\}$ in $\mathcal{B}$ such that

- (i) $s < t$ implies $A(s) \subset A(t)$, and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

REMARKS. In particular for the range of φ we have $\{\varphi(B)|B \in \mathcal{B}\}$ is all of the interval $[0, 1]$. It should also be noted that the sets $A(t)$ obtained depend heavily on the mean φ being considered.

PROOF. Let A_0 and $A_1 \in \mathcal{B}$ with $A_0 \subset A_1$ and let $\lambda \in [\varphi(A_0), \varphi(A_1)]$. Since $(\lambda - \varphi(A_0)) \in [0, \mu_\varphi(U_{A_1} \sim U_{A_0})] = [0, \varphi(A_1) - \varphi(A_0)]$ there exists a Borel set B in $\Delta(A)$ with $B \subset U_{A_1} \sim U_{A_0}$ and $\mu_\varphi(B) = \lambda - \varphi(A_0)$. Letting $E = U_{A_0} \cup B$ we have $U_{A_0} \subset E \subset U_{A_1}$ and $\mu_\varphi(E) = \lambda$.

Given $\varepsilon > 0$, by the regularity of μ_φ and using the facts that U_{A_0} is compact and U_{A_1} is open, we can find a compact E_1 and an open E_2 with $U_{A_0} \subset E_1 \subset E \subset E_2 \subset U_{A_1}$ and $\mu_\varphi(E_2 \sim E_1) < \varepsilon$. Since E_1 is compact and E_2 open, using the fact that $\{U_B|B \in \mathcal{B}\}$ is a base for the topology, we can find $B \in \mathcal{B}$ with $E_1 \subset U_B \subset E_2$ so $|\varphi(B) - \lambda| < \varepsilon$. Since $A_0 \subset B \subset A_1$ this implies $\{\varphi(B)|B \in \mathcal{B}, A_0 \subset B \subset A_1\}$ is dense in $[\varphi(A_0), \varphi(A_1)]$.

Let $Q_n = \{k/2^n | k = 0, 1, \dots, 2^n\}$ and $Q = \bigcup_{n=1}^\infty Q_n$. For $t \in Q$ we want to define $A(t) \in \mathcal{B}$ such that $\{A(t)|t \in Q\}$ satisfies (i) and (ii). We proceed inductively by defining $A(0) = \emptyset$, $A(1) = S$ and assuming that we have $\{A(t)|t \in Q_n\}$ satisfying the required conditions. For $t \in Q_{n+1} \cap Q_n$ we let $A'(t) = A(t)$. If $t \in Q_{n+1} \sim Q_n$ let $t_1, t_2 \in Q_n$ be the maximum and minimum elements of Q_n for which $t_1 < t < t_2$. Choose sequences $\alpha_n \uparrow t$ and $\beta_n \downarrow t$ in (t_1, t_2) . Since $\{\varphi(B)|B \in \mathcal{B}, A(t_1) \subset B \subset A(t_2)\}$ is dense in $[t_1, t_2]$ we can find $C_1 \in \mathcal{B}$ with $A(t_1) \subset C_1 \subset A(t_2)$ and $\beta_2 < \varphi(C_1) < \beta_1$. Since $(\alpha_1, \alpha_2) \subset [t_1, \varphi(C_1)]$ we can find $D_1 \in \mathcal{B}$ with $A(t_1) \subset D_1 \subset C_1$ and $\alpha_1 < \varphi(D_1) < \alpha_2$. Since $(\beta_3, \beta_2) \subset [\varphi(D_1), \varphi(C_1)]$ there exists $C_2 \in \mathcal{B}$ with $D_1 \subset C_2 \subset C_1$ and $\beta_3 < \varphi(C_2) < \beta_2$. Continuing in this manner we obtain sequences $C_i \downarrow$ and $D_i \uparrow$ in $\mathcal{B}$ with $A(t_1) \subset D_i \subset C_j \subset A(t_2)$ for all i, j and such that $\alpha_i < \varphi(D_i) < \alpha_{i+1}$, $\beta_{i+1} < \varphi(C_i) < \beta_i$ for all i . If we let $A'(t) = \bigcup_{i=1}^\infty D_i \in \mathcal{B}$ we have $D_i \subset A'(t) \subset C_j$ for all i, j which implies $\varphi(A'(t)) = t$ and also $A'(t_1) \subset A'(t) \subset A'(t_2)$. In this manner we obtain a new collection $\{A'(t)|t \in Q_{n+1}\}$ in $\mathcal{B}$ which satisfies (i) and (ii) and extends the collection $\{A(t)|t \in Q_n\}$. By induction we have a collection $\{A(t)|t \in Q\}$ with $\varphi(A(t)) = t$ for all $t \in Q$ and if $t_1, t_2 \in Q$ with $t_1 < t_2$ then $t_1, t_2 \in Q_n$ for some n so $A(t_1) \subset A(t_2)$.

For $t \in [0, 1]$ let $A(t) = \bigcap \{A(s)|s \in Q, t \leq s\}$. Since Q is countable, $A(t) \in \mathcal{B}$ and if $t_0 < t_1$ we have $A(t_0) \subset A(t_1)$. Also since $s_1 = \varphi(A(s_1)) \leq \varphi(A(t)) \leq \varphi(A(s_2)) = s_2$ for all $s_1, s_2 \in Q$ with $s_1 \leq t \leq s_2$, the density of Q in $[0, 1]$ implies $\varphi(A(t)) = t$ so the collection satisfies the required conditions.

The same argument gives us, as well, the following

LEMMA 2.1B. Let $(X, \mathcal{S}, \mu)$ be a measure space. Let φ be a mean on

$L^\infty(X)$ for which the corresponding probability measure μ_φ on $\Delta(L^\infty(X))$ is continuous. Then there exists a family $\{A(t) | t \in [0, 1]\}$ in $\mathcal{S}$ such that

- (i) $s < t$ implies $A(s) \subset A(t)$, and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

3. Orbits in $\Delta(A)$. Let A be a left invariant function algebra on a semigroup S . If $\psi \in \Delta(A)$ then the left orbit of ψ

$$O(\psi) = \{L_t^* \psi \mid t \in S\}$$

is a subset of $\Delta(A)$ where L_t^* denotes the adjoint of the left translation operator L_t . In order to apply Lemma 2.1A we will be interested in the case where $O(\psi)$ is infinite for all $\psi \in \Delta(A)$. In this case any LIM on A will have the property that μ_φ is a continuous measure on $\Delta(A)$.

LEMMA 3.1A. *Let S be an infinite Borel subsemigroup of a locally compact group G and A the algebra of bounded Borel measurable functions on S . Then $O(\psi)$ is infinite for all $\psi \in \Delta(A)$.*

PROOF. First note that for $\psi_1 \neq \psi_2$ in $\Delta(A)$ we can find $B \in B(S)$ with $\psi_1(B) \neq \psi_2(B)$. Let $C = tB$ where $t \in S$. Since S is a Borel set in G we have $B(S) = \{B \in B(G) \mid B \subset S\}$ and since $t \cdot B(G)$ is contained in $B(G)$ for all $t \in G$, this implies that $C \in B(S)$. Thus $\chi_C \in A$ with $L_t \chi_C = \chi_B$ so

$$L_t^* \psi_1(C) = \psi_1(L_t \chi_C) = \psi_1(B) \neq \psi_2(B) = \psi_2(L_t \chi_C) = L_t^* \psi_2(C)$$

and the mapping $L_t^* : \Delta(A) \rightarrow \Delta(A)$ is one-to-one.

If $O(\psi)$ is finite for some $\psi \in \Delta(A)$ then we can find $t_1, t_2, \dots, t_n \in S$ such that

- (i) $L_{t_i}^* \psi \neq L_{t_j}^* \psi$ for $i \neq j$, and
- (ii) for any $t \in S$ we have $L_t^* \psi = L_{t_i}^* \psi$ for some i .

If we set $\varphi = (\sum_{i=1}^n L_{t_i}^* \psi) / n$ it is easily checked that φ is a LIM on A and so, by restriction, also a LIM on $LUC(S)$ which is a closed subspace of A . However Theorem 3 of [4] shows that for infinite S , $LUC(S)$ has no LIM in the convex hull of the multiplicative means. This contradiction shows that $O(\psi)$ must be infinite for all $\psi \in \Delta(A)$.

The same argument can be used to prove as well

LEMMA 3.1B. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group G . Then $O(\psi)$ is infinite for all $\psi \in \Delta(L^\infty(S))$.*

4. Main results.

THEOREM 4.1. *Let S be a semigroup, $\mathcal{B}$ a left invariant σ -field of subsets of S and A the algebra of bounded $\mathcal{B}$ -measurable functions on S .*

If $O(\psi)$ is infinite for all $\psi \in \Delta(A)$ then for any LIM φ on A there exists a family $\{A(t) | t \in [0, 1]\}$ in $\mathcal{B}$ such that

- (i) $s < t$ implies $A(s) \subset A(t)$, and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

PROOF. This result follows from Lemma 2.1A if we can show that μ_φ is continuous. Choose $\psi \in \Delta(A)$ and $t \in S$. Note that if $L_t^* \psi \in U_B = \{\omega \in \Delta(A) | \omega(B) = 1\}$ where $B \in \mathcal{B}$ then $\psi \in U_{t^{-1}B}$ and since

$$\mu_\varphi(U_B) = \varphi(B) = \varphi(L_t \chi_B) = \varphi(\chi_{t^{-1}B}) = \mu_\varphi(U_{t^{-1}B})$$

we have $\mu_\varphi(\{\psi\}) \leq \mu_\varphi(U_B)$ whenever $L_t^* \psi \in U_B$. The regularity of μ_φ and the fact that $\{U_B | B \in \mathcal{B}\}$ is a base for the open sets in $\Delta(A)$ implies that $\mu_\varphi(\{\psi\}) \leq \mu_\varphi(\{L_t^* \psi\})$.

Since $O(\psi)$ is infinite, for any integer n we can find $t_1, t_2, \dots, t_n$ in S with $L_{t_i}^* \psi \neq L_{t_j}^* \psi$ for $i \neq j$. Thus

$$1 = \mu_\varphi(\Delta(A)) \geq \mu_\varphi\left(\bigcup_{i=1}^n \{L_{t_i}^* \psi\}\right) = \sum_{i=1}^n \mu_\varphi(\{L_{t_i}^* \psi\}) \geq n \cdot \mu_\varphi(\{\psi\})$$

and since n is arbitrary, this implies that $\mu_\varphi(\{\psi\}) = 0$ so μ_φ is continuous.

Combining this result with Lemma 3.1A we obtain

THEOREM 4.2A. *Let S be an infinite Borel subsemigroup of a locally compact group G . If A is the algebra of bounded Borel measurable functions on S and φ is a LIM on A then there exists a family $\{A(t) | t \in [0, 1]\}$ of Borel subsets of S such that*

- (i) $s < t$ implies $A(s) \subset A(t)$ and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

As a corollary to Theorem 4.2A we obtain a slightly stronger version of Chou's result in [1].

COROLLARY 4.2.1. *Let G be an infinite discrete group and let φ be a LIM on $m(G)$ (the bounded real valued functions on G with the sup norm). Then there exists a family $\{A(t) | t \in [0, 1]\}$ of subsets of G such that*

- (i) $s < t$ implies $A(s) \subset A(t)$, and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

The method of proof used in Theorem 4.1 will also work in the case where S is a Borel subsemigroup of positive Haar measure in a locally compact group and $A = L^\infty(S)$ (strictly speaking not a function algebra). Combining this with Lemma 3.1B we obtain

THEOREM 4.2B. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group G . If φ is a LIM on $L^\infty(S)$ then there*

exists a family $\{A(t) | t \in [0, 1]\}$ of Borel subsets of S such that

- (i) $s < t$ implies $A(s) \subset A(t)$, and
- (ii) $\varphi(A(t)) = t$ for all $t \in [0, 1]$.

5. Comments. Let S be an infinite, left amenable, right cancellation discrete semigroup and $A = m(S)$ (the bounded real valued functions on S). If S contains no element of infinite order then Lemma 4 of [2] shows that S is a group so, given any LIM φ on A , Theorem 4.2A gives us a nested family of subsets of S on which φ attains all values in the interval $[0, 1]$. On the other hand if $t \in S$ has infinite order, then for any integer n , Lemma 1 of [3] gives us a partition of S into disjoint subsets $A_1, A_2, \dots, A_n$ which are such that $tA_i \subset A_{i+1}$ for $1 \leq i \leq n-1$ and $tA_n \subset A_1$. If $\psi \in \Delta(A)$ we can assume that $\psi \in U_{A_1}$ and if $0 \leq k \leq n-1$ we have $t^k A_1 \subset A_{k+1}$ so $L_{t^k} \chi_{A_{k+1}} \geq L_{t^k} \chi_{t^k A_1} \geq \chi_{A_1}$ implies $1 \geq \psi(L_{t^k} \chi_{A_{k+1}}) \geq \psi(A_1) = 1$.

Thus $L_{t^k}^* \psi \in U_{A_{k+1}}$ for $k=0, 1, \dots, n-1$. But $U_{A_i} \cap U_{A_j} = \emptyset$ for $i \neq j$ so $\psi, L_t^* \psi, \dots, L_{t^{(n-1)}}^* \psi$ is a set of n distinct elements in $O(\psi)$. Since n was arbitrary this implies that $O(\psi)$ is infinite for all $\psi \in \Delta(A)$ and Theorem 4.1 gives us, for any LIM φ on $m(S)$, a nested family of subsets of S on which φ attains all values in $[0, 1]$. This result is weaker than Theorem 3 in [2]. There Granirer is able to find such a nested family of left almost convergent sets (i.e. $\varphi(A(t)) = t$ for any LIM φ on $m(S)$).

It would be interesting to determine whether the range of a LIM on $L^\infty(S)$ is attained on the left almost convergent Borel subsets of S , for any semigroup S .

REFERENCES

1. C. Chou, *On a conjecture of E. Granirer concerning the range of an invariant mean*, Proc. Amer. Math. Soc. **26** (1970), 105-107. MR **41** #5519.
2. E. Granirer, *On the range of an invariant mean*, Trans. Amer. Math. Soc. **125** (1966), 384-394. MR **34** #4390.
3. ———, *Extremely amenable semigroups*, Math. Scand. **17** (1965), 177-197. MR **33** #5760.
4. E. Granirer and A. Lau, *Invariant means on locally compact groups*, Illinois J. Math. **15** (1971), 249-257.
5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. 1: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #158.

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