ON SUBSETS WITH ASSOCIATED COMPACTA IN DISCRETE ABELIAN GROUPS

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ABSTRACT. Let Γ be a discrete abelian group. We prove that every non-Sidon set in Γ contains F , a non-Sidon set with the property that for every $\varepsilon > 0$ and compact set $K \subset \hat{\Gamma}$ with nonempty interior, there exists a finite set $\Lambda(\varepsilon, K) \subset F$, so that

$$\sup_{x \in K} |p(x)| \geq (1 - \varepsilon) \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(\hat{\Gamma}).$$

In this paper we demonstrate that the necessary condition for Sidon sets in discrete, torsion free abelian groups, obtained by M. Déchamps-Gondim in [2], (Theorem A), fails in a strong way to be sufficient (Theorem B).

In what follows below, Γ is a discrete abelian group, and $\hat{\Gamma} = G$. We refer to Chapters 1 and 2 of [6] for standard notation and facts. $E \subset \Gamma$ is a Sidon set if there exists $\alpha > 0$ so that $\|p\|_{\infty} \geq \alpha \sum_{\gamma \in \Gamma} |\hat{p}(\gamma)|$, for all trigonometric polynomials $p \in C_E(G) = \{f \in C(G) : \hat{f}(\gamma) = 0 \text{ for } \gamma \notin E\}$.

DEFINITION 1. $E \subset \Gamma$ is said to satisfy property (D) if for every $K \subset G$, a compact set with nonempty interior, there exists a finite set $\Lambda \subset E$, and $C > 0$, so that

$$\sup_{x \in K} |p(x)| \geq C \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(G).$$

DEFINITION 2. $E \subset \Gamma$ is said to satisfy property (D1) if for every $\varepsilon > 0$ and compact set $K \subset G$ with nonempty interior, there exists a finite set $\Lambda(\varepsilon, K) = \Lambda \subset E$, so that

$$\sup_{x \in K} |p(x)| \geq (1 - \varepsilon) \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(G).$$

Detailed studies of properties (D) and (D1) and their relationships to Sidon sets appear in [3], [4], and [5].

THEOREM A (M. DÉCHAMPS-GONDIM). *If Γ is torsion free, and $E \subset \Gamma$ is a Sidon set, then E satisfies property (D).*

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THEOREM B. *Let Γ be a discrete abelian group and $E \subset \Gamma$ a non-Sidon set. Then there exists a non-Sidon set $F \subset E$ so that F satisfies property (D1).*

Let Γ be a countable group. Then, $\hat{\Gamma} = G$ is a compact metrizable group, and therefore there exists D , a countable dense subgroup of G . Consider D as a discrete abelian group, and let $\phi: \Gamma \rightarrow \hat{D}$ be the natural injective map: $(\phi(\gamma), d) = (\gamma, d)$, for $\gamma \in \Gamma$ and $d \in D$. We say that $F \subset \hat{D}$ is a Sidon set if E is a Sidon set in $(\hat{D})_d$, $\hat{D}$ with the discrete topology. Our proof of Theorem B is based on the following two lemmas, for whose proof we refer to Lemmas 2.2 and 2.3 in [1].

LEMMA 3. *Let Γ , D , and ϕ be as above. Then $E \subset \Gamma$ is a Sidon set if and only if $\phi(E)$ is Sidon.*

LEMMA 4. *Let $E \subset \Gamma$ be a non-Sidon set. Then there exists a non-Sidon set $F \subset E$, so that $\overline{\phi(F)}$ (closure in $\hat{D}$) is a countable set with one limit point.*

PROOF OF THEOREM B. We first assume that Γ is a countable group, and we let D and ϕ be as above. By Lemma 4, we choose a non-Sidon set $F \subset E$ so that $\phi(F)$ accumulates only at $x_0 \in \hat{D}$, and without loss of generality we assume $x_0 = 0$. Suppose $\varepsilon > 0$ is given, and $K \subset G$ is a compact neighborhood of 0. By the density of D in G , and by the compactness of G , we select $S = \{d_j\}_{j=1}^N \subset D$, so that $\{K + d_j\}_{j=1}^N$ is a covering for G . By a theorem of Wiener (e.g., cf. [1, 1.3]), we find V , a neighborhood of 0 in $\hat{D}$, so that if $\mu \in M(V)$ then $|\hat{\mu}(d) - \hat{\mu}(d')| < (\varepsilon/2) \|\hat{\mu}\|_\infty$, whenever $d - d' \in S$. Let $\Lambda \subset F$ be a finite set so that $\phi(F \setminus \Lambda) \subset V$. Suppose p is a trigonometric polynomial in $C_{F \setminus \Lambda}(G)$, $p(g) = \sum_{j=1}^M a_j(\gamma_j, g)$, and $\|p\|_\infty = 1$. We can consider $\hat{p}$ as a discrete measure on $\hat{D}$: $\hat{p} = \mu = \sum_{j=1}^M a_j \delta_{\phi(\gamma_j)}$, and $\|\hat{\mu}\|_\infty = 1$. We find $g_0 \in G$ so that $|\sum_{j=1}^M a_j(\gamma_j, g_0)| = 1$, and let $g \in K$ be so that $g_0 = g + d_{j_0}$, where $d_{j_0} \in S$. By the density of D in G , we find $d' \in K \cap D$, so that

$$\left| 1 - \sum_{j=1}^M a_j(\gamma_j, d' + d_{j_0}) \right| < \varepsilon/2.$$

But, by the choice of V ,

$$\left| \sum_{j=1}^M a_j(\phi(\gamma_j), d') - \sum_{j=1}^M a_j(\phi(\gamma_j), d' + d_{j_0}) \right| < \varepsilon/2,$$

and therefore,

$$1 - \varepsilon \leq \left| \sum_{j=1}^M a_j(\gamma_j, d') \right| \leq \sup_{x \in K} |p(x)|.$$

If Γ is a discrete abelian group, and $E \subset \Gamma$ is a non-Sidon set, then there exists a countable non-Sidon set $E' \subset E$. It easily follows that $S \subset gp(E')$

is a Sidon set in Γ if and only if S is a Sidon set in $gp(E')$. Therefore, there exists a non-Sidon set $F \subset E$, so that F satisfies property (D1) with respect to $gp(E')^\wedge$. Let $\varepsilon > 0$ be given, and K be any compact neighborhood of 0 in G . The canonical map $\tau: G \rightarrow G/gp(E')^\perp$ is an open map, and therefore we can find a finite set $\Lambda \subset F$, so that

$$\sup_{x \in \tau(K)} |g(x)| \geq (1 - \varepsilon) \|g\|_\infty, \quad \text{for all } g \in C_{F \setminus \Lambda}(gp(E')^\wedge).$$

But, if $p \in C_{F \setminus \Lambda}(G)$, then $p(x+y) = p(x)$, for all $x \in G$, $y \in gp(E')^\perp$, and

$$\sup_{x \in K} |p(x)| \geq (1 - \varepsilon) \|p\|_\infty. \quad \square$$

REMARK. We say that a set $E \subset \Gamma$ has infinite pace, if given any finite set $F \subset \Gamma$, $F + E \cap E$ is a finite set. It is a consequence of a theorem of Wiener (cf. proof of Theorem B) that if $E \subset \Gamma$ satisfies property (D1) and G is connected, then E has infinite pace. We do not know whether the converse is true.

ADDED IN PROOF. Professor J.-P. Kahane indicated to us that the converse is false: There exists $E \subset \mathbb{Z}$ so that E has infinite pace, and E violates property (D).

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