K_3 OF A RING IS H_3 OF THE STEINBERG GROUP

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ABSTRACT. A proof of the result of the title is given using standard results of algebraic topology.

The result that $K_3(A) = H_3(\text{St}(A), \mathbb{Z})$ was suspected by many writers (e.g. [6, p. 207]) after Milnor gave his definition of K_2 of a ring (always unital and associative). The first proof, to my knowledge, was given by Quillen (unpublished) who based the result on Dror's homology Postnikov decomposition of a space with perfect fundamental group. The proof we offer here is extraordinary in its simplicity.

We recall a basic construction of Quillen ([4], [5]).

THEOREM. *If X is a based connected CW-complex and $E \leq \pi_1 X$ is a perfect subgroup of the fundamental group, then there is a map $f: X \rightarrow X^+$ such that*

(1) $\pi_1(f): \pi_1 X \rightarrow \pi_1 X^+$ is the quotient map of $\pi_1 X$ modulo the normal closure of E , and

(2) *If L is any $\pi_1 X^+$ module, then the map $H_*(X, L) \rightarrow H_*(X^+, L)$ is an isomorphism, where L denotes the corresponding local systems of coefficients.*

If one applies this construction to the Eilenberg-Mac Lane space $B_{\text{Gl}(A)} = K(\text{Gl}(A), 1)$ and subgroup $\mathcal{E}(A) \leq \text{Gl}(A)$, one has a space $B_{\text{Gl}(A)}^+$. One sets $K_i(A) = \pi_i(B_{\text{Gl}(A)}^+)$ for $i \geq 1$. This definition is known to recover the usual K_1 and K_2 of a ring [5].

LEMMA 1. $B_{\mathcal{E}(A)}^+$ is the universal cover of $B_{\text{Gl}(A)}^+$, where $B_{\mathcal{E}(A)} \rightarrow B_{\text{Gl}(A)}^+$ is Quillen's construction applied to $E = \mathcal{E}(A)$.

The proof is actually contained in the proof of Quillen's Theorem (cf. [5]).

PROPOSITION 2. *If $C \rightarrow E \rightarrow G$ is a central extension of groups where E is perfect, then the fibre of the map $B_E^+ \rightarrow B_G^+$ is B_C . Here B_E^+ and B_G^+ are Quillen's construction applied to the perfect groups E and G respectively.*

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PROOF. Look at the diagram

$$\begin{array}{ccccc} B_C & \longrightarrow & B_E & \longrightarrow & B_G \\ \downarrow f & & \downarrow & & \downarrow \\ F & \longrightarrow & B_E^+ & \longrightarrow & B_G^+ \end{array}$$

where F is the fibre of the map $B_E^+ \rightarrow B_G^+$. The spectral sequence comparison theorem [3, p. 356], implies $f: B_C \rightarrow F$ is a homology isomorphism (the actions in the Serre spectral sequences are trivial because the first row arises from a central extension and B_G^+ is simply connected in the second row). Since C is abelian, B_C is a simple space. But F is a nilpotent space, as one sees by a simple argument involving the homotopy exact sequence (cf. [1, 7.2]). It follows from Dror's generalization of the Whitehead Theorem [1, 4.3], that f is a homotopy equivalence.

Recall now that $\text{St}(A)$ is the universal central extension of $\mathcal{C}(A)$ ([2], [6]), and one has the central extension

$$K_2(A) \rightarrow \text{St}(A) \twoheadrightarrow \mathcal{C}(A).$$

In addition, $H_1(\text{St}(A), \mathbb{Z}) = H_2(\text{St}(A), \mathbb{Z}) = 0$ [2].

COROLLARY 3. $K_3(A) = H_3(\text{St}(A), \mathbb{Z})$.

PROOF. By Proposition 2, the fibre of the map $B_{\text{St}(A)}^+ \rightarrow B_{\mathcal{C}(A)}^+$ is $B_{K_2(A)} = K(K_2(A), 1)$. Thus, $\pi_i(B_{\text{St}(A)}^+) = \pi_i(B_{\mathcal{C}(A)}^+)$ for $i \geq 3$. But $B_{\text{St}(A)}^+$ is 1-connected and, by Quillen's Theorem, $H_1(B_{\text{St}(A)}^+) = H_2(B_{\text{St}(A)}^+) = 0$. Thus the Hurewicz Theorem gives

$$H_3(\text{St}(A), \mathbb{Z}) = H_3(B_{\text{St}(A)}^+) = \pi_3(B_{\text{St}(A)}^+).$$

On the other hand, by Lemma 1,

$$\pi_i(B_{\mathcal{C}(A)}^+) = \pi_i(B_{\text{Gl}(A)}^+) = K_i(A) \quad \text{for } i \geq 2.$$

Combining, we get $K_3(A) = H_3(\text{St}(A), \mathbb{Z})$, which completes the proof.

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