ON DIRECT PRODUCTS OF REGULAR p -GROUPS

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ABSTRACT. We prove that, for each prime p , there exists a regular p -group $H(p)$ with the property that, if G is a regular p -group and $G \times H(p)$ is regular, then the derived group of G has exponent p . This provides a strong converse to a theorem of Grün.

Introduction. It has long been known that the direct product of regular p -groups need not be regular. The first example was due to H. Wielandt and may be found in [2, III, 10.3]. In the positive direction, however, Grün [3] has shown that if G is a regular p -group whose derived group has exponent p , then $G \times H$ is regular for every regular p -group H . The purpose of this note is to prove a strong converse to this theorem. We will prove:

THEOREM. *For each prime p , there exists a regular p -group $H(p)$ with the following property:*

If G is a regular p -group and $G \times H(p)$ is also regular, then the derived group of G has exponent dividing p .

We recall that a finite p -group G is regular, if, whenever $g, h \in G$, there exists an element d of the derived group of the group generated by g and h such that $(gh)^p = g^p h^p d^p$. If, in addition, G' has exponent p , then $(gh)^p = g^p h^p$ for all g, h belonging to G and so G is p -abelian in the sense of Baer [1]. Conversely, if G is p -abelian, then G is evidently regular and it follows from standard results on regular groups (see, for example, [2, III, 10]), that G' has exponent dividing p . Hence, recalling the quoted theorem of Grün, we have

COROLLARY. *Let G be a regular p -group. Then $G \times H$ is regular, for each regular p -group H , if and only if G is p -abelian.*

As regular 2-groups are abelian, the Theorem is trivial if $p=2$ and so we will henceforth assume that p is an odd prime. We denote the commutator $g^{-1}h^{-1}gh$ of elements g and h of a group G by $[g, h]$ —and similarly for higher commutators—and the derived group of G by G' . Also, (g, h) will denote an element of the direct product $G \times H$, where $g \in G$ and $h \in H$. All groups considered in this note are finite.

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Proof of the Theorem. We begin by constructing the group $H(p)$. We have some freedom of choice in this; in fact it will suffice that our group have the following properties:

- (i) $H(p)$ can be generated by two elements a and b .
- (ii) $H(p)$ is regular but not p -abelian.
- (iii) Every commutator, in $H(p)$, of weight 3 or more has order dividing p .
- (iv) $(ab)^p = a^p b^p$ and $[a, b]^p \neq 1$.

Our construction is essentially due to Paul M. Weichsel [4], who proves a similar theorem under the restriction that G be metabelian. We repeat it here largely for convenience (and because it involves a slight twist on that construction), but refer to [4] for fuller details. Let A denote the direct product $\langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_{p-1} \rangle$, where $\langle a_i \rangle$ is a cyclic group, of order p^2 if $i=1$ or 2 and of order p if $i \geq 3$. Let t denote $(p-1)/2$ and let b denote the automorphism of A defined by:

$$\begin{aligned} b: a_i &\rightarrow a_i a_{i+1} & (1 \leq i < p-1), \\ b: a_{p-1} &\rightarrow a_{p-1} a_2^{p^t}. \end{aligned}$$

Then the group, $H(p)$, that we require, is the split extension of A by $\langle b \rangle$. It can be verified, either by direct calculation or by reference to [4], that b is, in fact, an automorphism of A and that the group $H(p)$ satisfies the required conditions, with $a_1 = a$. In particular, property (iv) is verified as follows:

$$(ab)^p = a^p b^p [b, a]^{p^t} [a, b, \dots, b] = a^p b^p$$

($p-1$ times)

since $[a, b, \dots, b] = [a, b]^{p^t}$.

Let G be a p -group which is regular but not p -abelian; we will show that $G \times H(p)$ is irregular. It suffices to assume that every proper subgroup and every proper homomorphic image of G is p -abelian—for, otherwise, we could take a section of G with these properties. We will now extract a few relevant properties of G .

As G is not p -abelian, there exist elements g and h of G such that $(gh)^p \neq g^p h^p$ and, by the minimality of G , these must generate G . Let M be a central subgroup of order p . By the minimality of G , G/M is p -abelian and so G'/M has exponent p . Thus, if g_1, h_1 and k are arbitrary elements of G , $[g_1, h_1]^p \in M$ and so $[[g_1, h_1]^p, k] = 1$. Hence, by a standard result on regular groups [2, III, 10.6], $[g_1, h_1, k]^p = 1$ and so every commutator of weight 3 or more has order dividing p . Thus, as G is regular, $(gh)^p = g^p h^p [g, h]^{pr}$, for some integer r . But $[g, h]^p \in M$ and so, as $(gh)^p \neq g^p h^p$, $[g, h]$ has order precisely p^2 and r is prime to p .

The proof of the theorem now follows very quickly. For, suppose that $G \times H(p)$ were regular, and let $x = (g, a)$ and $y = (h, b)$. As G and $H(p)$

both have the property that commutators of weight 3 or more have order dividing p , $G \times H(p)$ also has this property. Hence $(xy)^p = x^p y^p [x, y]^{ps}$ for some integer s . But,

$$\begin{aligned}(xy)^p &= (gh, ab)^p = ((gh)^p, (ab)^p) \\ &= (g^p h^p [g, h]^{pr}, a^p b^p) = x^p y^p ([g, h]^{pr}, 1).\end{aligned}$$

Thus,

$$([g, h]^{pr}, 1) = [x, y]^{ps} = ([g, h]^{ps}, [a, b]^{ps}),$$

and so $[g, h]^{pr} = [g, h]^{ps}$ and $[a, b]^{ps} = 1$. But $[a, b]^p \neq 1$ and therefore $p \nmid s$. It follows that $[g, h]^{pr} = 1$ —a contradiction which completes the proof of the theorem.

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