QUOTIENTS OF PROXIMITY SPACES

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ABSTRACT. A characterization of the quotient proximity is given. It is used to find necessary and sufficient conditions for every proximity map on a space to be a topological quotient map. It is shown that a separated proximity space is compact iff every p -map on X with separated range is a proximity quotient map.

Introduction. In 1959 Katetov [3] introduced proximity quotient maps. They have since been studied by Nachman [6], Poljakov ([7] and [8]), and Stone [9]. Although there are characterizations of proximity quotient maps in the literature [6], the only explicit formulation of the quotient proximity the author knows of is due to A. H. Stone, whose work appears in [10]. We give another characterization and use it to study mapping properties of proximity spaces.

Our notation will follow [10]. In particular, $A \subset \subset B$ will mean $A \not\delta (X-B)$ and proximity maps will be called p -maps. (X, δ) will always denote a (not necessarily separated) proximity space. Given a completely regular space X , δ_0 will represent the fine proximity: $A \delta_0 B$ iff there is some $f \in C^*(X)$ such that $f(A)=0$ and $f(B)=1$.

1. Characterization.

1.1 DEFINITION. If γ and δ are two proximities on a set X , γ is said to be finer than δ if $A \gamma B$ implies $A \delta B$. This will be written $\delta < \gamma$.

1.2 DEFINITION. Let f be a function from a proximity space (X, δ) onto a set Y . The quotient proximity is the finest proximity on Y such that f is a p -map. When Y has the quotient proximity, f will be called a p -quotient map.

1.3 THEOREM (STONE [9]). *The quotient proximity is given by: $C \subset \subset D$ iff for each binary rational $s \in [0, 1]$, there is some $C_s \subseteq Y$ such that $C_0 = C$, $C_1 = D$ and $s < t$ implies $f^{-1}(C_s) \subset \subset f^{-1}(C_t)$.*

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1.4 DEFINITION. Let f be a function from a proximity space (X, δ) onto a set Y . Define $A \delta^* B$ in Y if there is a function $g: Y \rightarrow I$ such that $g(A)=0$, $g(B)=1$ and $g \circ f$ is a p -map.

1.5 LEMMA. δ^* is a proximity.

PROOF. Clearly the axioms P1-P3 [10] hold and $A \delta^* B$ or $A \delta^* C$ easily imply $A \delta^* (B \cup C)$. Let $A \delta^* B$ and $A \delta^* C$. We must show $A \delta^* (B \cup C)$. Let g and h map Y to I such that $g(A)=0$, $g(B)=1$, $h(A)=0$, $h(C)=1$ and $g \circ f$ and $h \circ f$ are p -maps. Then $(g \cdot h)(A)=0$, $(g \cdot h)(B \cup C)=1$, and since the product of bounded real-valued p -maps is a p -map [3], $(g \cdot h) \circ f = (g \circ f) \cdot (h \circ f)$ is a p -map. Therefore, $A \delta^* (B \cup C)$.

Now, let $A \delta^* B$. Then there is a $g: Y \rightarrow I$ such that $g(A)=0$, $g(B)=1$ and $g \circ f$ is a p -map. The "strong axiom", P5, follows if g is a p -map. Let $C \delta^* D$. If $\text{cl}(g(C)) \cap \text{cl}(g(D)) = \emptyset$, there is a p -map $h: I \rightarrow I$ such that $h(\text{cl}(g(C)))=0$ and $h(\text{cl}(g(D)))=1$. But then $h \circ g \circ f$ is a p -map and $C \delta^* D$ —contradiction.

1.6 THEOREM. δ^* is the quotient proximity.

PROOF. First, $f: (X, \delta) \rightarrow (Y, \delta^*)$ is a p -map. For if $A \delta^* B$, there is some $h: Y \rightarrow I$ such that $h(A)=0$, $h(B)=1$ and $h \circ f$ is a p -map. It follows that $f^{-1}h^{-1}(0) \delta f^{-1}h^{-1}(1)$ and therefore $f^{-1}(A) \delta f^{-1}(B)$.

Now, if δ' is another proximity on Y such that $f: (X, \delta) \rightarrow (Y, \delta')$ is a p -map, then $A \delta' B$ implies there is a p -map $h: (Y, \delta') \rightarrow I$ such that $h(A)=0$ and $h(B)=1$. Clearly, $A \delta^* B$ and hence δ^* is the finest proximity on Y for which f is a p -map.

1.7 COROLLARY. A function g satisfying Definition 1.4 is continuous relative to the quotient topology on Y .

In the sequel δ^* will always denote the quotient proximity and $\zeta(\delta^*)$ the topology generated by δ^* .

2. In this section we consider the question of when a proximity quotient map is a (topological) quotient map.

2.1 EXAMPLE. Let X be the nonnegative real line with the usual proximity, identify n and $1/n$ for each positive integer n , and let Y be the resulting set. Consider $U = [0, \frac{1}{2}) \cup \bigcup_{n \geq 2} \{(n-1/n, n+1/n)\}$. Since $U = f^{-1}f(U)$, where f is the natural map, $f(U)$ is open in the quotient topology on Y . However, if $f(0) \subset \subset f(U)$, there must be some V such that $f(0) \subset \subset V \subset \subset f(U)$. Now, $\langle f(n) \rangle$ converges to $f(0)$, so there is an $N > 2$ with $f(n) \in V$ for all $n \geq N$. Thus, $\{f(n)\}_{n \geq N} \delta^* Y - f(U)$. But clearly, $\{n\}_{n \geq N} \delta (X - U)$,

so that $\{f(n)\}_{n \geq N} \delta^* Y - f(U)$. Therefore, $f(U)$ is not a $\zeta(\delta^*)$ -neighborhood of $f(0)$ and $\zeta(\delta^*)$ is not the quotient topology even though $f^{-1}(y)$ is finite for all $y \in Y$.

2.2 THEOREM. *Let $f: (X, \delta) \rightarrow (Y, \delta^*)$ be a 1-1 p -quotient map. Then (1) $A \delta^* B$ iff $f^{-1}(A) \delta f^{-1}(B)$, (2) f is a p -isomorphism and (3) the topology induced by δ^* is the quotient topology.*

PROOF. (1) Let $f^{-1}(A) \delta f^{-1}(B)$ and let $g: X \rightarrow I$ be a p -map such that $g(f^{-1}(A)) = 0$ and $g(f^{-1}(B)) = 1$. If $h: Y \rightarrow I$ is defined by $h(y) = g(f^{-1}(y))$, then $h \circ f = g$ and so is a p -map. Hence, $A \delta^* B$. (2) and (3) follow easily from (1).

2.3 THEOREM. *If δ_0 is the fine proximity on X and $f: (X, \delta_0) \rightarrow (Y, \delta^*)$ is a p -quotient map, then $\zeta(\delta^*)$ is the quotient topology iff the quotient topology is completely regular.*

PROOF. Let the quotient topology, α , be completely regular and let δ' be any proximity compatible with α . Since δ_0 is the fine proximity, $f: (X, \delta_0) \rightarrow (Y, \delta')$ is a p -map so that $\delta' < \delta^*$. Hence, $\alpha = \zeta(\delta') \subseteq \zeta(\delta^*)$. The opposite inclusion follows because every p -map is continuous.

Necessity is obvious.

2.4 THEOREM. *Let (X, δ) be a proximity space. Then every proximity quotient of X generates the quotient topology iff (1) δ is the fine proximity and (2) every (topological) quotient is completely regular.*

PROOF. Sufficiency follows from Theorem 2.3. For the necessity, assume every proximity quotient generates the quotient topology. If $\delta \neq \delta_0$, there are two sets A and B such that $A \delta B$ but $A \delta_0 B$. That is, A and B are functionally separated. It follows that there is an open set U such that $\bar{A} \subseteq U$ and $U \cap B = \emptyset$. Let Y be the set formed by identifying $\bar{A}$ to a point, and give Y the quotient proximity. Then, if f is the natural map, $f(U)$ is an open neighborhood of $f(\bar{A})$ in the quotient topology. But $\bar{A} \delta B$ implies $f(\bar{A}) \delta^* f(B)$, so $f(\bar{A}) \in \text{cl}_{\zeta(\delta^*)} f(B)$. Clearly, $f(U)$ cannot be a $\zeta(\delta^*)$ -neighborhood of $f(\bar{A})$ —contradiction. Condition (2) above easily holds.

REMARK. Exactly the same proof will show that every separated proximity quotient generates the quotient topology iff (1') $\delta = \delta_0$ and (2') every T_2 quotient is completely regular. MacDonald and Willard [5] have recently considered the problem of characterizing the spaces X such that every T_2 quotient of X is regular.

3. Mapping properties.

3.1 THEOREM. *Let (X, δ) be a separated proximity space. Then every p -map on X with separated range is a p -quotient map iff (X, δ) is compact.*

PROOF. Assume (X, δ) is not compact. Then there is a cluster π without a point. (See [4].) Let p be any point in X and define $\pi' = \pi \cup \pi_p$, where π_p is the cluster of all A such that $A \delta p$. Finally, define $A \delta' B$ iff either $A \delta B$ or both A and $B \in \pi'$.

We claim that δ' is a proximity on X . Axioms (P1)–(P3) of [10] are easily verified. To prove (P4), observe that π' , as the union of two clusters, inherits the following property of clusters:

(i) $B \cup C \in \pi'$ iff $B \in \pi'$ or $C \in \pi'$.

Now, let $A \delta' (B \cup C)$. If $A \delta (B \cup C)$, we are done, so assume $A \not\delta (B \cup C)$. Then we must have $A \in \pi'$ and $(B \cup C) \in \pi'$. It follows from (i) that $A \delta' B$ or $A \delta' C$. The reverse implication in (P4) also follows from (i).

To prove (P5), let $A \not\delta' B$. Then either $A \notin \pi'$ or $B \notin \pi'$. By symmetry we need only consider the case where $A \notin \pi'$. Then, $A \not\delta B$, $A \not\delta p$, and since $A \notin \pi$, $A \not\delta P$ for some $P \in \pi$. Thus, there exist disjoint sets U and V such that $A \not\delta (X - U)$ and $(B \cup P \cup \{p\}) \not\delta (X - V)$. But $(X - V) \notin \pi$ since $(X - V) \not\delta P$ and $P \in \pi$, and $(X - V) \notin \pi_p$ since $(X - V) \not\delta p$, so $(X - V) \notin \pi'$. Since A and $(X - V)$ are not in π' , $A \not\delta' (X - U)$ and $(B \cup P \cup \{p\}) \not\delta' (X - V)$, and (P5) is satisfied.

Note that δ' is easily separated and $\delta' < \delta$. It follows from our assumption that $i: (X, \delta) \rightarrow (X, \delta')$ is a one-to-one p -quotient map, and so, by Theorem 2.2, $\delta = \delta'$. This contradicts the definition of δ' ; therefore, (X, δ) is compact.

Conversely, assume (X, δ) is compact and let f be a p -map from X onto a separated proximity space (Y, δ') . Then the quotient topology is $T_{3\frac{1}{2}}$ and δ is the fine proximity on X . It follows from Theorem 2.3 that $\zeta(\delta^*)$ is the quotient topology α . Since $f: (X, \zeta(\delta)) \rightarrow (Y, \zeta(\delta'))$ is a continuous function on a compact set, $\zeta(\delta') = \alpha$. But since (Y, α) is compact and T_2 , it has a unique compatible proximity [10]; hence, δ' is equivalent to δ^* , the quotient proximity.

REMARK. It is well known that a $T_{3\frac{1}{2}}$ space X is locally compact iff it has a minimal compatible proximity. If we consider a separated proximity on a set X to be minimal separated if $\delta' < \delta$ and δ' separated imply $\delta' = \delta$, then we have the following

3.2 COROLLARY. *Let (X, δ) be a separated proximity space. Then (X, δ) is compact iff δ is minimal separated on X .*

3.3 THEOREM. *Let Y be any completely regular space and δ' any compatible proximity. Then δ' is the fine proximity on Y iff for all proximity*

spaces (X, δ) and all p -maps $f: (X, \delta) \rightarrow (Y, \delta')$ onto Y such that $\zeta(\delta')$ is the quotient topology induced by f , then f is a p -quotient map.

PROOF. Let $\delta' = \delta_0$, the fine proximity on Y , and let $f: (X, \delta) \rightarrow (Y, \delta_0)$ be a p -map onto Y such that $\zeta(\delta_0)$ is the quotient topology α . Then $\delta_0 < \delta^*$ where δ^* is the quotient proximity. If $A \not\delta^* B$ there is some $g: Y \rightarrow I$ such that g is continuous relative to $\alpha = \zeta(\delta')$ and $g(A) = 0$, $g(B) = 1$ and $g \circ f$ is a p -map. Therefore $A \not\delta_0 B$ so that $\delta_0 = \delta^*$ and f is p -quotient.

For the converse, consider $i: (Y, \delta_0) \rightarrow (Y, \delta')$ and apply Theorem 2.2.

If $f: (X, \delta) \rightarrow (Y, \delta^*)$ is a p -quotient map, we shall write $\delta^* = \delta/f$.

3.4 COROLLARY. Let (X, δ) be a proximity space and $(Y, \delta/f)$ a p -quotient such that the quotient topology is completely regular. Then δ/f is the fine proximity on the quotient topology iff $\delta/f = \delta_0/f$, where δ_0 is the fine proximity on X .

PROOF. Sufficiency is easy while necessity follows from the above theorem. $\square$

Poljakov ([7] and [8]) defined a p -map to be p -open if $A \subset \subset B \Rightarrow f(A) \subset \subset f(B)$. The proof of the next theorem is routine and thus omitted.

3.5 THEOREM. Let f map a proximity space (X, δ) onto a proximity space (Y, δ') . Then f is p -open iff $f(A) \delta' B$ implies $A \delta f^{-1}(B)$.

3.6 THEOREM. Let (X, δ) be a proximity space such that every p -quotient map on X is p -open. Then every p -quotient generates the quotient topology. Consequently, δ is the fine proximity on X .

PROOF. If X has a p -quotient $(Y, \delta/f)$ which does not generate the quotient topology, then f is not open, hence not p -open—contradiction. $\square$

There do not seem to be reasonable sufficient conditions on X such that every p -quotient on X is p -open. For example, if $X = [0, 2]$ and $[\frac{1}{2}, \frac{3}{2}]$ is identified to a point, the natural map is p -quotient but not open. A related topological problem is to characterize the topological spaces X such that every topological quotient map on X is open. See MacDonald and Willard [5] for similar problems.

Proximity bi-quotient maps will be considered in a forthcoming paper.

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