PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 1, March 1973

A PROPERTY OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS

F. T. HOWARD

ABSTRACT. Let $g(n)$ be a rational function of n whose denominator is divisible by the same power of 2 for each n and let $a_1, a_2, \cdots$ be any sequence of rational numbers such that for $n>1$, $a_n=g(n)(a_1a_{n-1}+a_2a_{n-2}+\cdots+a_{n-1}a_1)$. In this paper we determine the exact power of 2 dividing the denominator of a_n for each n and prove congruences (mod 4) and (mod 8).

1. **Introduction.** When dealing with special sequences of rational numbers, we often want to answer the following question: For a given prime p, what is the exact power of p dividing the numerator or denominator of the nth number in the sequence? In general this is a difficult question to answer, especially, if nothing is known about the numbers other than a recurrence formula expressing the nth number of the sequence in terms of the previous numbers. Another generally difficult problem is to prove congruences for the numbers (mod p^k), $k\geqq 1$. For both of these problems it appears that the case $p=2$ is usually easier to deal with than $p>2$.

In this paper we consider any sequence $a_1, a_2, \cdots$ of rational numbers such that, for $n>1$,

$$a_n = g(n)\sum_{k=1}^{n-1} a_k a_{n-k},$$

where $g(n)$ is a rational function of n which is divisible by exactly the same power of 2 for each n. Examples of such numbers are $a_n=B_{2n}/(2n)!$ where B_{2n} is the $2n$th Bernoulli number and $a_n=\binom{2n-2}{n-1}/n$. We shall determine the exact power of 2 dividing the denominator (or numerator if 2 does not divide the denominator) of a_n and prove congruences (mod 4) and (mod 8). Examples are discussed in §6. We note that one of these examples is discussed in [4] and [3] and that a conjecture in [4] is proved in this paper.

Presented to the Society, November 20, 1971; received by the editors September 21, 1971.

AMS (MOS) subject classifications (1970). Primary 10A35, 10A40, 10A10.

Key words and phrases. Nonlinear difference equation, recurrence formula, rational function, congruence, Bernoulli number, Rayleigh function.

2. **Preliminaries.** The notation of the introduction and the definitions and lemmas of this section will be used in the subsequent sections.

DEFINITION 2.1. Define $\theta(n)$ as the exponent of the highest power of 2 dividing the denominator of a_n, $n=1, 2, \cdots$. If $\theta(n)$ is negative then $-\theta(n)$ is the exponent of the highest power of 2 dividing the numerator of a_n.

DEFINITION 2.2. Define t as the exponent of the highest power of 2 dividing the denominator of $g(n)$, $n=2, 3, \cdots$. If t is negative then $-t$ is the exponent of the power of 2 dividing the numerator of $g(n)$.

DEFINITION 2.3. Let n be a positive integer and let m be the number of nonzero terms in the base 2 expansion of n. Define $f(n, s)$ as the number of positive integers $r \leqq n/2$ such that the number of nonzero terms in the base 2 expansion of r plus the number of nonzero terms in the base expansion of $n-r$ is equal to $m+s$.

The following lemmas are proved in [3] and are necessary for the proofs of the congruences in this paper.

LEMMA 2.1. *If there are m nonzero terms in the base 2 expansion of n then* $f(n, 0)=2^{m-1}-1$.

LEMMA 2.2. *Let* $n=2^{e_1}+\cdots+2^{e_m}$, $m>1$, $e_i-e_{i+1}>1$, *for q terms* e_i, $e_i-e_{i+1}=1$, *for $m-q-1$ terms. Then* $f(n, 1)=(q+1)2^{m-2}$ *if* $e_m \geqq 1$ *and* $f(n, 1)=q2^{m-2}$ *if* $e_m=0$.

LEMMA 2.3. *Let* $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>s>0$, *for* $i=1, \cdots, m-1$ *and* $e_m \geqq s$. *Then*

$$f(n, s) = \sum_{j=1}^{m} \binom{m}{j} \binom{s-1}{j-1} \alpha(j) 2^{s+m-2j-1},$$

where $\alpha(j)=2$ *if* $s+m=2j$, $\alpha(j)=1$ *if* $s+m>2j$.

3. **The power of 2 dividing a_n.**

THEOREM 3.1. *Let $n \geqq 1$ and let m be the number of nonzero terms in the base 2 expansion of n. Then* $\theta(n)=n\theta(1)+1+(n-1)t-m$.

PROOF. The proof is by induction on n. The theorem is true for $n=1$. Assume it is true for $1, \cdots, n-1$ and let $x=n\theta(1)+1+(n-1)t-m$. We have

$$2^x a_n = 2^x g(n) \sum_{k=1}^{[n/2]} \beta(k) a_k a_{n-k}, \tag{3.1}$$

where $\beta(k)=2$ if $k \neq n/2$, $\beta(k)=1$ if $k=n/2$. For any k we consider $2^{x-t}\beta(k)a_k a_{n-k}$. Suppose there are h nonzero terms in the base 2 expansion of k and w nonzero terms in the base 2 expansion of $n-k$.

Case 1. $m \neq 1$. We have by our induction hypothesis

$$2^{x-t}\beta(k)a_k a_{n-k} = 2^{\theta(k)}a_k \cdot 2^{\theta(n-k)}a_{n-k} \cdot 2^{h+w-m}$$

or

$$= (2^{\theta(n/2)}a_{n/2})^2 \cdot 2^{m-1} \quad \text{if } k = n/2.$$

Hence every term on the right side of (3.1) is congruent to 0 (mod 2) except those for which $h+w=m$. By Lemma 2.1 there are $2^{m-1}-1$ such terms, an odd number.

Case 2. $m=1$. In this case every term on the right side of (3.1) is congruent to 0 (mod 2) except when $k=n/2$. Hence in both Cases 1 and 2 we have $2^x a_n \equiv 1 \pmod 2$.

4. **Congruences** (mod 4). We must assume that a_1 and $g(n)$ satisfy one of the following three sets of congruences for all n. We let r stand for either 1 or 3.

(4.1) $$2^t g(n) \equiv 2^{\theta(1)}a_1 \equiv r \pmod 4,$$

(4.2) $$2^t g(n) \equiv (-1)^n r \pmod 4, \qquad 2^{\theta(1)}a_1 \equiv -r \pmod 4,$$

(4.3) $$2^t g(n) \equiv (-1)^n r \pmod 4, \qquad 2^{\theta(1)}a_1 \equiv r \pmod 4.$$

An example of (4.1) is $a_1=1$, $g(n)=1$, $n=2, 3, \cdots$.

An example of (4.2) is $a_1=1/(2a+4)$, $g(n)=2/(a+2n)$, where a is odd.

An example of (4.3) is $a_1=1/12$, $g(n)=-1/(2n+1)$.

THEOREM 4.1. *Let* $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>1$, *for* q *terms* e_i, $e_i-e_{i+1}=1$, *for* $m-1-q$ *terms* e_i $(i=1, \cdots, m-1)$. *Then if* (4.1) *or* (4.2) *holds, we have, for* $n>1$,

$$2^{\theta(n)}a_n \equiv (-1)^q r \pmod 4.$$

PROOF. The proof is by induction on n. The theorem is true for $n=2$ since by equation (3.1) and Theorem 3.1

$$2^{\theta(2)}a_2 \equiv r(2^{\theta(1)}a_1)^2 \equiv r \pmod 4.$$

Assume the theorem is true for $2, \cdots, n-1$ and let n satisfy the hypotheses of the theorem.

Case 1. $m=1$. In this case we have $n=2^{e_1}$, $e_1>0$, $q=0$ and by equation (3.1), Theorem 3.1 and our induction hypothesis we have

$$2^{\theta(n)}a_n \equiv r(2^{\theta(n/2)}a_{n/2})^2 \equiv r \pmod 4.$$

Case 2. $m=2$. In this case $n=2^{e_1}+2^{e_2}$ and we verify directly, as in Case 1, that the theorem holds in the four possible cases: $e_1-e_2>1$, $e_2>0$; $e_1-e_2>1$, $e_2=0$; $e_1-e_2=1$, $e_2>0$; $e_1-e_2=1$, $e_2=0$.

The proofs in the final three cases are quite similar to each other. We shall state the cases and then prove Case 5.

Case 3. $m>2$ and in the base 2 expansion of n there are two terms 2^{e_i}, 2^{e_j} such that $e_{i-1}-1>e_i>e_{i+1}+1$ and $e_{j-1}-1>e_j>e_{j+1}+1$. We note that e_i could be either e_1 or e_m, in which case the conditions are $e_1>e_2+1$ or $e_{m-1}-1>e_m$.

Case 4. $m>2$ and in the base 2 expansion of n there is a term 2^{e_i} such that $e_i-e_{i+1}=1$, $e_{i-1}>1+e_i$ (if $i\neq 1$) and $e_{i+1}>1+e_{i+2}$ (if $i+1\neq m$).

Case 5. $m>2$ and in the base 2 expansion of n there is a term 2^{e_i} such that $e_i-e_{i+1}=1$, $e_{i+1}-e_{i+2}=1$ and $e_{i-1}-e_i>1$ (if $i\neq 1$).

To prove Case 5 we shall use the letters h and w as they were used in the proof of Theorem 3.1. We shall also assume that (4.1) holds, since the proof is very similar when (4.2) holds. We first want to consider those terms on the right side of equation (3.1) for which $h+w=m$. Let $E=\{e_1, \cdots, e_m\}$. We know that if $h+w=m$ then the exponents of 2 in the base 2 expansions of k and $n-k$ must be elements of E. Consider any distribution of the elements of $E-\{e_i, e_{i+1}\}$ between k and $n-k$ for which there is at least one element assigned to k and at least one element assigned to $n-k$. Then there are four possibilities for the assignments of e_i and e_{i+1} and we have $2^{\theta(k)}a_k\cdot 2^{\theta(n-k)}a_{n-k}$ is congruent to $(-1)^z$, $z=1$ or -1 if e_i, e_{i+1} and e_{i+2} are all assigned together; $(-1)^{z+1}$ if e_{i+1} and e_{i+2} are assigned together, e_i assigned differently; $(-1)^{z+1}$ if e_i and e_{i+1} are assigned together, e_{i+2} assigned differently; $(-1)^{z+2}$ if e_i and e_{i+2} are assigned together, e_{i+1} assigned differently.

Notice that the sum of all such terms $2^{\theta(k)}a_k\cdot 2^{\theta(n-k)}a_{n-k}$ is congruent to 0 (mod 4).

If all the elements of $E-\{e_i, e_{i+1}\}$ are assigned to k (or $n-k$) then by our induction hypothesis $2^{\theta(k)}a_k\cdot 2^{\theta(n-k)}a_{n-k}$ is congruent to $(-1)^q$ if e_i and e_{i+1} are assigned together (and hence separated from e_{i+2}); $(-1)^q$ if e_{i+1} is assigned with e_{i+2} and e_i is assigned differently; $(-1)^{q+1}$ if e_i is assigned with e_{i+2} and e_{i+1} is assigned differently.

Therefore we have

$$\begin{aligned} 2^{\theta(n)}a_n &\equiv r[2(-1)^q+(-1)^{q+1}]+2rf(n,1) \pmod 4 \\ &\equiv (-1)^q r \pmod 4 \end{aligned}$$

since, by Lemma 2.2, $f(n, 1)\equiv 0 \pmod 2$.

The next theorem is proved in a similar way. We omit the proof.

THEOREM 4.2. *Let* $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>1$, *for* q *terms* e_i, $e_i-e_{i+1}=1$, *for* $m-1-q$ *terms* e_i $(i=1, \cdots, m-1)$. *Then if* (4.3) *holds, we have, for* $n>1$,

$$2^{\theta(n)}a_n\equiv (-1)^{q+n}r \pmod 4.$$

5. **Congruences** (mod 8). In this section we must assume that a_1 and $g(n)$ satisfy one of the following three sets of congruences, where r is either 1, 3, 5 or 7.

$$2^t g(n) \equiv 2^{\theta(1)} a_1 \equiv r \pmod 8, \tag{5.1}$$

$$2^t g(n) \equiv (-1)^n r \pmod 8, \qquad 2^{\theta(1)} a_1 \equiv -r \pmod 8, \tag{5.2}$$

$$2^t g(n) \equiv (-1)^n r \pmod 8, \qquad 2^{\theta(1)} a_1 \equiv r \pmod 8. \tag{5.3}$$

We shall also make use of Lemma 2.3, Theorem 3.1, Theorem 4.1 and Theorem 4.2.

THEOREM 5.1. *Suppose any one of* (5.1), (5.2) *or* (5.3) *holds for all* n. *If* $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>2$, *for* $i=1, \cdots, m-1$ *and* $e_m \geqq 2$, *then*

$$\begin{aligned} 2^{\theta(n)} a_n &\equiv 7r \pmod 8 \quad \textit{if } m \textit{ is even},\\ &\equiv 5r \pmod 8 \quad \textit{if } m \textit{ is odd}. \end{aligned}$$

PROOF. The proof is by induction on m. We first verify that it is true for $m=1, 2, 3$. If $n=2^{e_1}$, then

$$2^{\theta(n)} a_n \equiv rf(n, 1) + 4rf(n, 2) \equiv 5r \pmod 8.$$

If $n=2^{e_1}+2^{e_2}$, then

$$2^{\theta(n)} a_n \equiv rf(n, 0) + 6rf(n, 1) + 4r(f(n, 2) - 1) + 2r \equiv 7r \pmod 8.$$

If $n=2^{e_1}+2^{e_2}+2^{e_3}$, then

$$2^{\theta(n)} a_n \equiv 3rf(n, 0) + 2rf(n, 1) + 4rf(n, 2) + 4r \equiv 5r \pmod 8.$$

Assume the theorem is true for $1, 2, 3, \cdots, m-1$ and $n=2^{e_1}+\cdots+2^{e_m}$. If m is even we have

$$2^{\theta(n)} a_n \equiv rf(n, 0) + 6rf(n, 1) + 4rf(n, 2) \equiv 7r \pmod 8.$$

If m is odd, we have

$$2^{\theta(n)} a_n \equiv 3rf(n, 0) + 2rf(n, 1) + 4rf(n, 2) \equiv 5r \pmod 8.$$

6. **Examples.**

EXAMPLE 6.1. Let $a_n=\binom{2n-2}{n-1}/n$. Then $a_1=1$ and (see [**6**, pp. 74–75])

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k}.$$

If $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>1$, for q terms e_i, then, by Theorem 3.1, $\theta(n)=1-m$. Therefore the exponent of the highest power of 2 dividing

$\binom{2n-2}{n-1}$ is $m+s-1$, where s is the exponent of the highest power of 2 dividing n. Furthermore,

$$2^{1-m-s}\binom{2n-2}{n-1} \equiv 2^{-s}n(-1)^q \pmod 4.$$

If $e_i-e_{i+1}>2$, $e_m \geqq 2$, then

$$\begin{aligned} 2^{1-m-s}\binom{2n-2}{n-1} &\equiv 2^{-s}7n \pmod 8, \quad \text{if } m \text{ is even,} \\ &\equiv 2^{-s}5n \pmod 8, \quad \text{if } m \text{ is odd.} \end{aligned}$$

We note that Wolstenholme in about 1880 pointed out that the highest power of 2 dividing $\binom{2n-1}{n}$ is $m-s-1$ where m is the number of nonzero terms in the base 2 expansion of $2n-1$ and s is the exponent of the highest power of 2 dividing n. Cesaro, Kummer, Van den Broeck and others considered this type of problem also [**2**, pp. 270–272].

EXAMPLE 6.2. Let $a_n=B_{2n}/(2n)!$ where B_{2n} is the $2n$th Bernoulli number, defined by

$$\frac{x}{e^x-1} = \sum_{r=0}^{\infty} B_r \frac{x^r}{r!}.$$

Then $a_1=1/12$ and (see [**7**, p. 146])

$$a_n = \frac{-1}{2n+1} \sum_{k=1}^{n-1} a_k a_{n-k}.$$

If $n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>1$, for q terms e_i, then we have $\theta(n)=1+2n-m$. Since $2n-m$ is the exponent of the highest power of 2 dividing $(2n)!$ [**2**, p. 263] we have the well-known result that if $n\geqq 1$, the denominator of B_{2n} is divisible by 2 but not by 4. Also, by Theorem 4.2,

$$2^{1+2n-m}B_{2n}/(2n)! \equiv (-1)^{q+n+1} \pmod 4.$$

We note that in [**1**] it is pointed out that

$$2B_{2n} \equiv 1 \pmod 4, \qquad n>1,$$

and in fact

$$2B_{2n} \equiv 1+4n \pmod{16}, \qquad n>2.$$

Thus it follows that, for $n>1$,

$$2^{m-2n}(2n)! \equiv (-1)^{q+n+1} \pmod 4.$$

EXAMPLE 6.3. Let $a_n = \sigma_{2n}(a/b)$, the Rayleigh function of argument a/b, a odd, b even (see [5]). Then $a_1 = b/4(a+b)$ and

$$a_n = \frac{b}{a+bn} \sum_{k=1}^{n-1} a_k a_{n-k}.$$

This example is discussed in [4] and [3]. We note that Theorem 4.1 proves a conjecture in [4], namely that if $b=(2k+1)2^t$, $t>0$, then

$$2^{\theta(n)}\sigma_{2n}(a/b) \equiv (-1)^q((2k+1)/a) \pmod 4.$$

REFERENCES

1. L. Carlitz, *A property of the Bernoulli numbers*, Amer. Math. Monthly **67** (1960), 1011–1012.
2. L. E. Dickson, *History of the theory of numbers*. Vol. 1, Publication No. 256, Carnegie Institution of Washington, Washington, D.C., 1919.
3. F. T. Howard, *A combinatorial problem and congruences for the Rayleigh function*, Proc. Amer. Math. Soc. **26** (1970), 574–578. MR **42** #1756.
4. ———, *A property of the Rayleigh function*, Proc. Amer. Math. Soc. **24** (1970), 719–723. MR **40** #7495.
5. N. Kishore, *The Rayleigh function*, Proc. Amer. Math. Soc. **14** (1963), 527–533. MR **27** #1633.
6. C. L. Liu, *Introduction to combinatorial mathematics*, McGraw-Hill, New York, 1968. MR **38** #3154.
7. N. E. Norlund, *Vorlesungen über Differenzrechnung*, Springer-Verlag, Berlin, 1924.

DEPARTMENT OF MATHEMATICS, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NORTH CAROLINA 27109