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DIFFERENTIABILITY OF THE METRIC PROJECTION IN FINITE-DIMENSIONAL EUCLIDEAN SPACE

EDGAR ASPLUND

ABSTRACT. The metric projection on a closed subset of a finite-dimensional Euclidean space is almost everywhere differentiable.

The main purpose of this short note is to point out that the answer to a question by Kruskal [3] is implicit in a famous theorem of A. D. Alexandrov [1] (of which a new proof has recently been given by Rešetnjak [4]) which says that each continuous convex function on $\boldsymbol{R}^n$ is almost everywhere twice differentiable. For $n=1$, this reduces to Lebesgue's theorem about the differentiability almost everywhere of a monotone function.

Using Alexandrov's theorem one can prove the following theorem which contains the answer to Kruskal's question.

THEOREM. *The metric projection on any closed subset of a finite-dimensional Euclidean space is almost everywhere differentiable.*

Consider $\boldsymbol{R}^n$ as provided with the standard Euclidean norm. For a closed $K \subset \boldsymbol{R}^n$ and an element $x \in \boldsymbol{R}^n$ let $p(x)$ be the nearest point in K to x. This may not be everywhere uniquely defined (as a matter of fact, this happens if and only if K is convex) but we make a selection and call the function $p: \boldsymbol{R}^n \to K$ so defined the *metric projection* on K. In Asplund [2, p. 42 et seq.], it is shown that the convex function f, defined and continuous on all of $\boldsymbol{R}^n$ by

$$f(x) = \sup\{(x, y) - \|y\|^2/2 \mid y \in K\} = \|x\|^2/2 - \inf_{y \in K} \|x - y\|^2/2,$$

has $p(x)$ for a differential at all points x where f is once differentiable. Moreover, at those points where f is not differentiable, $p(x)$ is an element of the subdifferential of f at x. These are all easy facts, and the details of the calculations can be found in the paper [2]. An obvious calculation along the same lines then shows that our theorem here is a consequence of Alexandrov's.

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The technique, used here and in [2], to represent a metric projection as the gradient of a convex function works only in Euclidean space. It would therefore be interesting to know for which finite-dimensional Banach spaces it is true that the metric projection on each closed subset is almost everywhere differentiable.

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THE INSTITUTE FOR OPTIMIZATION AND SYSTEMS THEORY, THE ROYAL INSTITUTE OF TECHNOLOGY, STOCKHOLM 70, SWEDEN

Current address: Department of Mathematics, Odense University, DK-5000 Odense, Denmark