ON WEAKLY FLAT 1-SPHERES

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ABSTRACT. With his cellularity criterion, McMillan has provided a characterization of those complementary domains of an $(n-1)$ -sphere in the n -sphere S^n that are open n -cells ($n \neq 4$). Paul F. Duvall has used the cellularity criterion to characterize those k -spheres Σ in S^n ($2 \leq k \leq n-3$) such that $S^n - \Sigma$ is homeomorphic to the complement of the standard k -sphere in S^n . The purpose of this note is to indicate how McMillan's methods can be reapplied to handle the case $k=1$ ($n \neq 4$) as well.

Following Duvall we say that a k -sphere Σ in S^n is *weakly flat* iff $S^n - \Sigma$ is homeomorphic to the complement of a standardly embedded k -sphere in S^n .

A compact nonseparating subset X of a manifold Y is said to have the global 1-alg property iff for each open set U containing X there exists an open set V of Y such that $X \subset V \subset U$ and, for the inclusion induced homomorphism

$$i_*: \pi_1(V - X, v) \rightarrow \pi_1(U - X, v), \quad v \in V - X,$$

$i_*(\pi_1(V - X, v))$ is an abelian subgroup of $\pi_1(U - X, v)$; equivalently, every loop in $V - X$ that is null-homologous in $V - X$ is null-homotopic in $U - X$.

For a k -sphere Σ in S^n ($2 \leq k \leq n-3$), Σ has the global 1-alg property iff Σ satisfies the cellularity criterion. Accordingly, the analogue of the results of McMillan [3, Theorem 4] and Duvall [2, Theorem 2.1] is the following theorem.

THEOREM 1. *A 1-sphere Σ in S^n ($n \geq 5$) is weakly flat if and only if Σ has the global 1-alg property.*

If Σ is weakly flat, it is easy to show that Σ has the global 1-alg property. We shall sketch the proof of the converse.

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LEMMA 1. *If a 1-sphere Σ in S^n ($n \geq 4$) has the global 1-*alg* property, then for each open set U containing Σ the pair $(U, U - \Sigma)$ is 2-connected.*

PROOF. Showing that $(U, U - \Sigma)$ is 0- and 1-connected is trivial. Since Σ is an ANR, there exists an open set V containing Σ such that Σ is a strong deformation retract of V in U (that is, there exists a homotopy $h_t: V \rightarrow U$ such that $h_0 = \text{identity}$, $h_1(V) = \Sigma$, and $h_t|_{\Sigma} = 1_{\Sigma}$ ($0 \leq t \leq 1$)). Thus, each map of the boundary of a 3-cell K into V can be extended to a map of K into U .

To establish the 2-connectedness of $(U, U - \Sigma)$, observe that elementary duality arguments imply that $H_1(N - \Sigma) = 0$ for each open n -cell N in S^n . Consequently there exists an open set W containing Σ and there exists $\delta > 0$ such that every loop L in $W - \Sigma$ satisfying $\text{diam } L < \delta$ is null-homologous in $V - \Sigma$. After restricting W further, if necessary, we can assume by the global 1-*alg* property that each such loop L is null-homotopic in $V - \Sigma$.

Let Δ^2 denote a 2-simplex. Consider a map $f: (\Delta^2, \text{Bd } \Delta^2) \rightarrow (U, U - \Sigma)$, and choose a triangulation T of Δ^2 having small mesh. Since Σ separates no open subset of S^n , there exists a map $g: (\Delta^2, \text{Bd } \Delta^2) \rightarrow (U, U - \Sigma)$ that is homotopic (rel $\text{Bd } \Delta^2$) to f in U and $g(T^{(1)}) \subset U - \Sigma$, where $T^{(1)}$ denotes the 1-skeleton of T . It is easy to do this so that $\text{diam } g(\tau) < \delta$ and $g(\tau) \subset W$ for each 2-simplex τ of T satisfying $g(\tau) \cap \Sigma \neq \emptyset$. We then obtain a map $g': \Delta^2 \rightarrow U - \Sigma$ by redefining g on the interiors of such 2-simplexes τ so that $g'(\tau) \subset V - \Sigma$. The remark at the end of the first paragraph implies that g' is homotopic (rel $\text{Bd } \Delta^2$) to g and, therefore, to f .

A tube in S^n is the image in S^n of $S^1 \times \Delta_{n-1}$, where Δ_{n-1} denotes an $(n-1)$ -simplex, under a PL-embedding.

LEMMA 2. *Let J denote a 1-sphere topologically embedded in S^n ($n \geq 5$). For each open set U containing J there exists an open set V containing J such that every finite $(n-3)$ -polyhedron in V is contained in a tube $T \subset U$.*

PROOF. By Lemma 2.7 of [1], for each open set U containing J there exists an open set V containing J such that a regular neighborhood T^* of a PL 1-sphere very close to J can be used to engulf, by pushing T^* through U , any finite $(n-3)$ -polyhedron in V by a tube.

LEMMA 3. *Let U be a neighborhood of a 1-sphere Σ in S^n ($n \geq 5$) such that Σ has the global 1-*alg* property. Then there exists a tube T such that $\Sigma \subset \text{Int } T \subset T \subset U$.*

PROOF. With Lemma 2 in place of McMillan's observations about CAR's, the argument proceeds like the one he gives for Theorem 1 of [3].

PROOF OF THEOREM 1. Suppose that Σ has the global 1-alg property. Lemma 2 implies the existence of a sequence of tubes $T_1, T_2, \dots$ such that $\text{Int } T_i \supset T_{i+1}$ and $\bigcap T_i = \Sigma$. We can assume that Σ is a strong deformation retract of T_2 in T_1 . For $i=1, 2, \dots$ define a map π_i of T_i to a center line C_i , the image in T_i of $S^1 \times \{pt\}$ in $S^1 \times \Delta_{n-1}$. Since $\pi_i|_\Sigma$ is homotopic in T_i to 1_Σ and T_i deformation retracts in T_1 to Σ ($i \geq 2$), it follows that $\pi_i|_\Sigma$ is a degree one map of Σ to C_i . Hence, $\pi_i|_\Sigma$ is homotopic in T_i to a homeomorphism of Σ to C_i .

Claim. $\text{Cl}(T_i - T_{i+1})$ is homeomorphic to $S^1 \times S^{n-2} \times I$ ($i \geq 2$).

There is no loss of generality in assuming $C_i \cap C_{i+1} = \emptyset$. By the remark above C_i and C_{i+1} are homotopic to Σ in T_i , and using general position techniques one can construct easily a PL annulus (homeomorphic to $S^1 \times I$) A in T_i such that $\text{Bd } A = C_i \cup C_{i+1}$. It follows that C_{i+1} is a spine of T_i , and T_{i+1} must be a spine of T_i as well. The claim is then a consequence of Corollary 4 to Theorem 8 in [6, Chapter 3].

The rest of the argument is elementary.

A similar result holds for 1-spheres in S^3 , but the knotting problem must be avoided.

THEOREM 2. *If a 1-sphere Σ in S^3 has the global 1-alg property, then there exists an open set W containing Σ such that $\text{Cl}(W) - \Sigma$ is homeomorphic to $S^1 \times S^1 \times [0, 1)$.*

Proof. By performing surgery near Σ one can show that for each open set V containing Σ there exists a polyhedral torus T in S^3 separating $S^3 - V$ and Σ . We shall prove that such tori sufficiently close to Σ bound solid tori (or tubes) near Σ . Since $H_1(S^3 - \Sigma) \neq 0$, there exists a simple closed curve J in $S^3 - \Sigma$ that is not contractible in $S^3 - \Sigma$. As before, for each open set U containing Σ there exists an open set V containing Σ such that Σ is a strong deformation retract of V in U . Hence, for a polyhedral torus T in V separating Σ and $S^3 - V$, kernel $(\pi_1(T) \rightarrow \pi_1(U))$ is nontrivial. Application of the loop theorem [4] then yields a polyhedral 2-cell D in U such that $\text{Bd } D = D \cap T$ is not contractible in T . Clearly the closure X of that component of $S^3 - T$ containing $\text{Int } D$ is a solid torus. If, in addition, $U \cap J = \emptyset$, then $\Sigma \subset X$ (implying $X \subset U$), for otherwise J would be contained in $X - D$ and, therefore, would be contractible in $S^3 - \Sigma$.

Hence, there exists a sequence of polyhedral solid tori $T_1, T_2, \dots$ such that $\text{Int } T_i \supset T_{i+1}$ and $\bigcap T_i = \Sigma$. Careful application of the generalized loop theorem [5] implies that for sufficiently large i there exists a polyhedral annulus A_i in $\text{Cl}(T_i - T_{i+1})$ such that one component of $\text{Bd } A_i$ is contained in $\text{Bd } T_i$, the other in $\text{Bd } T_{i+1}$, and neither component of $\text{Bd } A_i$ is contractible in the appropriate $\text{Bd } T_i$. Invoking the global 1-alg

property, one can show that $T_i - (A_i \cup T_{i+1})$ has abelian (infinite cyclic) fundamental group, and with this fact one can demonstrate that $\text{Cl}(T_i - T_{i+1})$ is homeomorphic to $S^1 \times S^1 \times I$.

THEOREM 3. *A 1-sphere Σ in S^3 is weakly flat iff Σ has the global 1-alg property and $\pi_1(S^3 - \Sigma)$ is infinite cyclic.*

Here the hypothesis regarding $\pi_1(S^3 - \Sigma)$, with the loop theorem [4], implies that a 1-sphere Σ satisfying the global 1-alg property is contained in an unknotted solid torus T such that $T - \Sigma$ is homeomorphic to $S^1 \times S^1 \times [0, 1)$.

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