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LOWER BOUNDS FOR SOLUTIONS OF HYPERBOLIC INEQUALITIES IN UNBOUNDED REGIONS[1]

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ABSTRACT. This paper considers C^2 solutions $u=u(t, x)$ of the differential inequality $|Lu|\leqq k_1(t, x)|u|+k_2(t, x)\|\nabla u\|$. The coefficients of the hyperbolic operator L depend on both t and x. Explicit lower bounds are given for the energy of u in a region of x-space expanding at least as fast as wave-fronts for L. These bounds depend on the asymptotic behavior of k_1, k_2, and the coefficients of L. They do not require boundary conditions on u.

1. **Introduction.** Let L be a hyperbolic operator of the form $Lu= A(t, x)u-u_{tt}$ where $A(t, x)$ denotes a second order uniformly elliptic operator whose coefficients depend on the time variable t as well as the spatial coordinates $x=(x_1, \cdots, x_N)$. Several authors **[1]**, **[3]**, **[4]**, **[5]**, **[6]** have considered the asymptotic behavior of solutions of the equation

$$Lu = F(t, x, u, \nabla u). \tag{1.1}$$

Because of both the time-dependence in $A(t, x)$ and the presence of u_t on the right side, one cannot expect all solutions of (1.1) to behave like solutions of the wave equation.

In **[4]**, the author discussed the asymptotic behavior of C^2 solutions of the inequality

$$|Lu| \leqq k_1(t, x)\,|u| + k_2(t, x)\,|\nabla u|. \tag{1.2}$$

Such an inequality arises from (1.1) if F is assumed Lipschitz in its last two arguments. The results of **[4]** establish a kind of unique continuation at infinity, e.g., if a solution of (1.2) decays fast enough inside a forward characteristic conoid for L, then it must vanish there. This paper sharpens **[4]** by providing explicit lower bounds for nonvanishing solutions.

The bounds are comparable to those found by Ogawa **[5]** for the inequality $\|Lu\|\leqq k(t)\,\|\nabla u\|$ where $\|\cdot\|$ denotes the L^2 norm on a domain

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in $\boldsymbol{R}^N$. Ogawa's bounds improved certain maximal rate of decay results of Protter [**6**].

Recently Bloom and Kazarinoff [**1**] have announced upper bounds on solutions of $Lu=0$ in expanding regions outside an obstacle.

The operator $A(t, \boldsymbol{x})$ we consider is defined by

$$A(t, \boldsymbol{x})u = \sum_{ij=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(t, \boldsymbol{x}) \frac{\partial u}{\partial x_j} .$$

We assume that the coefficients a_{ij} are C^1 functions on the half-space $\mathscr{H}=\boldsymbol{R}^+ \times \boldsymbol{R}^N$ with $a_{ij}=a_{ji}$. Further, we assume that there are positive constants m and M such that

$$m^2 \leqq \sum_{ij=1}^{N} a_{ij}(t, \boldsymbol{x})\xi_i\xi_j \leqq M^2 \tag{1.3}$$

for all $(t, \boldsymbol{x}) \in \mathscr{H}$ and all unit vectors ξ in $\boldsymbol{R}^N$. Thus the bilinear form

$$((b, c)) = b_0c_0 - \sum_{ij=1}^{N} a_{ij}(t, \boldsymbol{x})b_ic_j$$

defines a Lorentz metric in $\mathscr{H}$.

We can interpret a solution u of (1.2) as a scalar disturbance in a time varying anisotropic medium occupying $\boldsymbol{R}^N$. We study the energy of u in a region of $\boldsymbol{x}$-space which expands at least as fast as wave-fronts for L. Let $S(T)$ be the region at time T; formally we consider $S(T)$ as a domain in the hyperplane $t=T$ in $\mathscr{H}$. As T increases, the $S(T)$ sweep out a region $D(0, \infty)=\bigcup \{S(T):T>0\}$ in $\mathscr{H}$.

We say that region $S(T)$ expands faster than light, or faster than wave-fronts for L, if the following two conditions are met: First, the boundaries $\partial S(T)$ sweep out a smooth hypersurface S' in $\mathscr{H}$, which is the lateral part of the boundary of $D(0, \infty)$. Second, the outer unit normal $\boldsymbol{n}=(n_0, n_1, \cdots, n_N)$ on $\partial D(0, \infty)$ satisfies $n_0<0$ and $((\boldsymbol{n}, \boldsymbol{n}))\geqq 0$ along S'. For example, for a fixed r the region $S(T)=\{(T, \boldsymbol{x}):|\boldsymbol{x}|\leqq MT+r\}$ expands faster than light. If $\boldsymbol{n}$ is negative characteristic on S', then the $\partial S(T)$ are an expanding wave-front for L.

Suppose that the region $S(T)$ expands at least as fast as light. If w is a C^2 function in $\mathscr{H}$, we discuss its size in terms of the energy integral

$$\mathscr{E}(w, T) = \int_{S(T)} \left(w^2 + w_t^2 + \sum_{ij=1}^{N} a_{ij}w_{,i}w_{,j} \right) dx.$$

This paper gives conditions under which a solution u of (1.2) will satisfy a lower bound of the form

$$\mathscr{E}(u, T) \geqq Ce^{-\gamma f(T)}\mathscr{E}(u, \tau) \tag{1.4}$$

for $T>\tau\geqq 0$. In (1.4), C and γ are positive constants, and f is a function which increases without bound as $T\to\infty$.

In particular we have the following results:

RESULT I. If $k_1(t, x)=O(t^{-2})$, $k_2(t, x)=O(t^{-1})$, and all $|(a_{ij})_t|=O(t^{-1})$, then (1.4) holds with $f(T)=\ln(T)$.

RESULT II. If k_1, k_2, and all $|(a_{ij})_t|$ are bounded, then (1.4) holds with $f(T) = T$.

RESULT III. If there is a constant $c>1$ such that $k_1(t, x)=O(t^{2c-2})$, $k_2(t, x)=O(t^{c-1})$, and all $|(a_{ij})_t|=O(t^{c-1})$, then (1.4) holds with $f(T)=T^c$.

For derivatives we use the notation $\partial w/\partial t=w_t$ and $\partial w/\partial x_i=w_{,i}$. The gradient ∇w is taken with respect to all $N+1$ variables; and $|\nabla w|^2=w_t^2+\sum_{i=1}^N w_{,i}^2$.

We introduce the quadratic form

$$P_{b,c}(\xi) = 2((\boldsymbol{b}, \xi))((\boldsymbol{c}, \xi)) - ((\boldsymbol{b}, \boldsymbol{c}))((\xi, \xi))$$

for vector fields $\boldsymbol{b}$, $\boldsymbol{c}$, and ξ on $\boldsymbol{R}\times\boldsymbol{R}^N$. As shown by Hörmander [2], this form is positive definite if $\boldsymbol{b}$ and $\boldsymbol{c}$ are positive timelike vectors. [A vector $\boldsymbol{d}=(d_0, d_1, \cdots, d_N)$ is positive timelike iff $d_0>0$ and $((\boldsymbol{d}, \boldsymbol{d}))>0$.] Notice that $P_{b,c}(\xi)$ is linear in $\boldsymbol{b}$ and $\boldsymbol{c}$. We use $\boldsymbol{h}$ to denote the timelike vector $\boldsymbol{h}=(1, 0, \cdots, 0)$ in $\boldsymbol{R}\times\boldsymbol{R}^N$. Since $P_{h,h}(\xi)=\xi_0^2+\sum_{i,j=1}^N a_{ij}\xi_i\xi_j$, $|\nabla w|$ and $(P_{h,h}(\nabla w))^{1/2}$ are equivalent.

The starting point for the basic estimates is the formula

$$\text{(1.5)}\quad \int_D 2\lambda w_t Lw = \int_D \left(P_{h,\nabla\lambda}(\nabla w) + \lambda\sum_{ij=1}^N (a_{ij})_t w_{,i}w_{,j}\right) - \int_{\partial D} \lambda P_{h,n}(\nabla w)$$

which is valid for any C^1 function $\lambda=\lambda(t, \boldsymbol{x})$ and any C^2 function $w=w(t, \boldsymbol{x})$ where D is a bounded domain with piecewise smooth boundary and $\boldsymbol{n}$ is the outer unit normal along ∂D. This formula follows directly from integration by parts.

2. **Proof of Result II.** In this section we consider (1.2) under the assumption that there are constants such that

$$\text{(2.1)}\qquad |(a_{ij})_t| \leqq K;\quad k_1(t, \boldsymbol{x}) \leqq K_1;\quad k_2(t, \boldsymbol{x}) \leqq K_2$$

in $\mathscr{H}$. We will prove Result II of the Introduction as Theorem 2.3. §3 will outline the slight modifications which adapt the proof to the hypotheses for Results I and II.

We start by developing the basic a priori inequality. Notice that no boundary conditions are imposed; the choice of domains and the properties of $P_{b,c}(\xi)$ combine to make boundary conditions unnecessary.

Let $S(T)$ be a region in $\boldsymbol{x}$-space expanding faster than light. Let $D(\tau, T)$ be the region in $\mathscr{H}$ swept out by the $S(t)$ for $\tau<t<T$; i.e.,

$$D(\tau, T) = \{(t, \boldsymbol{x}) : \boldsymbol{x} \in S(t) \text{ and } \tau < t < T\}.$$

Then the boundary $\partial D(\tau, T)$ is composed of three smooth pieces: $S(T)$, $S(\tau)$, and the lateral portion along S'. The outer unit normal $\boldsymbol{n}=(n_0, n_1, \cdots, n_N)$ on $\partial D(\tau, T)$ is equal to $\boldsymbol{h}$ on $S(T)$ and to $-\boldsymbol{h}$ on $S(\tau)$.

Suppose v is a C^2 function and D is one of the regions $D(\tau, T)$ for $0 \leqq \tau < T$. We develop a weighted L^2 estimate for v and ∇v in D in terms of Lv. To do this we introduce an auxiliary function $w = e^{\alpha t} v$ for α a positive parameter. Computation shows that

$$e^{\alpha t} Lv = Lw + 2\alpha w_t - \alpha w^2.$$

Using the elementary inequality $(X+Y+Z)^2 \geqq 2Y(X+Z)$ we get

$$e^{2\alpha t}|Lv|^2 \geqq 2(2\alpha w_t)(Lw - \alpha^2 w).$$

For $\beta > 0$, we multiply through by $e^{\beta t}$ and then integrate over D to obtain

$$\iint\limits_D e^{\beta t} e^{2\alpha t} |Lv|^2 \geqq 2\alpha \iint\limits_D 2 e^{\beta t} w_t Lw - 2\alpha^3 \iint\limits_D e^{\beta t} (w^2)_t. \tag{2.2}$$

Integration by parts and the properties of n on ∂D give us

$$\begin{aligned} -\iint\limits_D e^{\beta t}(w^2)_t &= \iint\limits_D \beta e^{\beta t} w^2 - \int_{\partial D} n_0 e^{\beta t} w^2 \\ &\geqq \beta \iint\limits_D e^{\beta t} w^2 + \int_{S(\tau)} e^{\beta t} w^2 - \int_{S(T)} e^{\beta t} w^2. \end{aligned} \tag{2.3}$$

The next two lemmas provide an estimate for the other integral on the right side of (2.2).

LEMMA 2.1. *Suppose $\beta m^2 \geqq 2KN$. Then*

$$\begin{aligned} 2\iint\limits_D e^{\beta t} w_t Lw \geqq \tfrac{1}{2}\beta \iint\limits_D e^{\beta t} P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) &- \int_{S(T)} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) \\ &+ \int_{S(\tau)} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w). \end{aligned} \tag{2.4}$$

PROOF. If $\lambda = e^{\alpha t}$ and $D = D(\tau, T)$, then formula (1.5) specializes to

$$2\iint\limits_D e^{\beta t} w_t Lw + \int_{\partial D} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{n}}(\nabla w) = \iint\limits_D e^{\beta t}\left\{\beta P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) + \sum_{ij=1}^{N} (a_{ij})_t w_{,i} w_{,j}\right\}. \tag{2.5}$$

By assumption (1.3) we have

$$P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) = w_t^2 + \sum_{ij=1}^{N} a_{ij} w_{,i} w_{,j} \geqq m^2 \sum_{i=1}^{N} (w_{,i})^2.$$

But we also have

$$\left|\sum (a_{ij})_t w_{,i} w_{,j}\right| \leqq \sum_{ij=1}^{N} |(a_{ij})_t|\,|w_{,i}|\,|w_{,j}|$$
$$\leqq K\left(\sum_{i=1}^{N} |w_{,i}|\right)^2 \leqq KN \sum_{i=1}^{N} (w_{,i})^2.$$

Since $\frac{1}{2}\beta m^2 \geqq KN$, we can conclude that

$$\beta P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) + \sum (a_{ij})_t w_{,i} w_{,j} \geqq \tfrac{1}{2}\beta P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w). \tag{2.6}$$

The properties of $\boldsymbol{n}$ on ∂D yield the inequality

$$\int_{\partial D} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{n}}(\nabla w) \leqq \int_{S(T)} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) - \int_{S(\tau)} e^{\beta t} P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w). \tag{2.7}$$

Using (2.6) and (2.7) to estimate terms in (2.5) we obtain the inequality of the lemma.

We now prove the basic a priori inequality.

THEOREM 2.2. *Suppose v is a C^2 function on $\bar{D}(0, \infty)$. If $\alpha > 1$, $\beta m^2 \geqq 2KN$, and $0 \leqq \tau < T$, then there are constants c_i such that*

$$\begin{aligned} &6\alpha^3 e^{(\beta+2\alpha)T} \mathscr{E}(v, T) + \iint_{D(\tau,T)} e^{(\beta+2\alpha)t} |Lv|^2 \\ &\qquad \geqq c_1 \alpha\beta \iint_{D(\tau,T)} e^{(\beta+2\alpha)t} (v^2 + |\nabla v|^2) + c_2 e^{(\beta+2\alpha)\tau} \mathscr{E}(v, \tau). \end{aligned} \tag{2.8}$$

PROOF. Let D denote $D(\tau, T)$ and let $w = e^{\alpha t} v$. Then we can combine inequalities (2.3), (2.4), and (2.2) to obtain

$$\begin{aligned} &\iint_D e^{(\beta+2\alpha)t} |Lv|^2 + 2\alpha \int_{S(T)} e^{\beta t} [P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) + \alpha^2 w^2] \\ &\qquad \geqq \alpha\beta \iint_D e^{\beta t} [P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) + 2\alpha^2 w^2] + 2\alpha \int_{S(\tau)} e^{\beta t} [P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) + \alpha^2 w^2]. \end{aligned} \tag{2.9}$$

From the expression of $P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w)$ in terms of v we can show that

$$P_{\boldsymbol{h},\boldsymbol{h}}(\nabla w) \leqq 2e^{2\alpha t} [\alpha^2 v^2 + P_{\boldsymbol{h},\boldsymbol{h}}(\nabla v)]$$

and

$$P_{h,h}(\nabla w) \geqq e^{2\alpha t}[-\tfrac{1}{2}\alpha^2 v^2 + \tfrac{1}{3}P_{h,h}(\nabla v)].$$

Thus (2.9) leads to the inequality

$$\iint_D e^{(\beta+2\alpha)t}|Lv|^2 + 2\alpha e^{(\beta+2\alpha)T}\int_{S(T)}[3\alpha^2 v^2 + P_{h,h}(\nabla v)]$$
$$\geqq \alpha\beta \iint_D e^{(\beta+2\alpha)t}[\tfrac{3}{2}\alpha^2 v^2 + \tfrac{1}{3}P_{h,h}(\nabla v)]$$
$$+ 2\alpha e^{(\beta+2\alpha)\tau}\int_{S(\tau)}[\tfrac{1}{2}\alpha^2 v^2 + \tfrac{1}{3}P_{h,h}(\nabla v)].$$

The desired inequality (2.8) now follows because of the hypothesis $\alpha > 1$, the equivalence of $P_{h,h}(\nabla v)$ with $|\nabla v|^2$, and the definition of $\mathscr{E}(v, T)$.

THEOREM 2.3. *Suppose u is a C^2 solution of (1.2) in the closure of $D(\tau, \infty)$. Then there are positive constants C and γ, independent of u, such that*

$$\mathscr{E}(u, T) \geqq Ce^{-\gamma T}\mathscr{E}(u, \tau)$$

for all $T > \tau$.

PROOF. Pick a fixed β so that $\beta m^2 \geqq 2KN$. Since u is a C^2 function we can apply Theorem 2.2. Let D denote $D(\tau, T)$. For $\alpha > 1$ and $T > \tau \geqq 0$, we have

$$6\alpha^3 e^{(\beta+2\alpha)T}\mathscr{E}(u, T) + \iint_D e^{(\beta+2\alpha)t}|Lu|^2$$
$$\geqq c_1\alpha\beta \iint_D e^{(\beta+2\alpha)t}(u^2 + |\nabla u|^2) + c_2 e^{(\beta+2\alpha)\tau}\mathscr{E}(u, \tau).$$

From (1.2) and (2.1) we find

$$\iint_D e^{(\beta+2\alpha)t}|Lu|^2 \leqq 2\iint_D e^{(\beta+2\alpha)t}(K_1^2 u^2 + K_2^2|\nabla u|^2).$$

Let $K_3 = \max\{K_1^2, K_2^2\}$. It follows that

$$6\alpha^3 e^{(\beta+2\alpha)T}\mathscr{E}(u, T) \geqq \iint_D e^{(\beta+2\alpha)t}(c_1\alpha\beta - 2K_3)(u^2 + |\nabla u|^2) + c_2 e^{(\beta+2\alpha)\tau}\mathscr{E}(u, \tau).$$

Setting $\gamma = \beta + 2\alpha$ for α sufficiently large we have

$$6\alpha^3 e^{\gamma T}\mathscr{E}(u, T) \geqq c_2 e^{\gamma\tau}\mathscr{E}(u, \tau).$$

The choice of C required for the theorem is now apparent.

COROLLARY. *If u is a nonnull solution of* (1.2) *in* $\bar{D}(\tau, \infty)$, *then* $\mathscr{E}(u, T)$ *cannot decay faster then* $e^{-\gamma T}$, *where* γ *is chosen as above.*

PROOF. Let C and γ be taken as in the theorem. If $\tau<\sigma<T$, then the proof of the theorem also shows that

$$\mathscr{E}(u, T) \geqq Ce^{-\gamma T}\mathscr{E}(u, \sigma).$$

So if $\lim_{T\to\infty} e^{\gamma T}\mathscr{E}(u, T)=0$, then $\mathscr{E}(u, \sigma)=0$ for all $\sigma>\tau$. This would show that $u\equiv 0$ in $\bar{D}(\tau, \infty)$.

3. **Remarks on Results I and III.** This section indicates how the proof of Theorem 2.3 is adapted to prove Results I and III. Again we consider a given family of regions $S(T)$ expanding faster than light.

THEOREM 3.1. *Suppose the coefficients of* (1.2) *satisfy the bounds*

$$|(a_{ij})_t| \leqq Kt^{-1}; \quad k_1(t, x) \leqq K_1t^{-2}; \quad k_2(t, x) \leqq K_2t^{-1} \tag{3.1}$$

in $\bar{D}(\tau, \infty)$ *for some* $\tau>0$. *Suppose u is a solution of* (1.2) *in* $\bar{D}(\tau, \infty)$. *Then there are constants C and γ, independent of u, such that*

$$\mathscr{E}(u, T) \geqq CT^{-\gamma}\mathscr{E}(u, \tau) \quad \textit{for all } T > \tau.$$

PROOF. The pattern of proof follows that of §2 exactly except that the weight functions $\lambda=e^{\alpha t}$ are replaced by $\lambda=\exp(\alpha\ln(t))=t^{\alpha}$. The decay conditions imposed on the coefficients of (1.2) by (3.1) are those required so that the weighted L^2 integral for Lu on $D(\tau, T)$ can be dominated by the terms in u^2 and $P_{h,h}(\nabla u)$ which arise from the a priori estimate analogous to Theorem 2.2.

THEOREM 3.2. *Suppose there is a constant $c>1$ such that the coefficients of* (1.2) *satisfy the bounds*

$$|(a_{ij})_t| \leqq Kt^{c-1}; \quad k_1(t, x) \leqq K_1t^{2c-2}; \quad k_2(t, x) \leqq K_2t^{c-1} \tag{3.2}$$

in some $\bar{D}(\tau, \infty)$. *Suppose that u satisfies* (1.2) *in* $\bar{D}(\tau, \infty)$. *There are constants C and γ, independent of u, such that*

$$\mathscr{E}(u, T) \geqq C\exp(-\gamma T^c)\mathscr{E}(u, \tau) \quad \textit{for all } T > \tau.$$

PROOF. Here again we follow the outline of §2 using this time the weight functions $\lambda=\exp(\alpha t^c)$ in place of $\lambda=e^{\alpha t}$.

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