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TRÈVES' IDENTITY[1]

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ABSTRACT. We give a very simple proof of the identity which Trèves obtained for partial differential operators with constant coefficients. Our proof uses very little information about the structure of the operator and applies to a much wider class of operators. The exact scope of our method is still undetermined. The identity can be applied to obtain a priori estimates, existence theorems, and construction of fundamental solutions.

In his thesis [1] Trèves proved a remarkable identity for differential operators with constant coefficients, which has many important implications. His proof uses strongly the representation of such an operator as a polynomial in the operators $X_j = -i\partial/\partial x_j$, and the commutation relations $[x_j, X_k] = i\delta_{jk}$. By verbal communication, he has subsequently found two other proofs, both of which also use detailed information about the structure of the operators. We give here a simple proof which uses very little about the structure of the operators. Our result may be more general, but we do not know of any interesting classes of operators satisfying our hypotheses and not covered by Trèves' result.

We consider linear operators defined on a linear subset $\mathscr{D}$ dense in a Hilbert space $\mathscr{H}$. Let x be a selfadjoint operator on $\mathscr{H}$ such that the unitary group $\exp(i\lambda x)$, $\lambda \in R$, leaves $\mathscr{D}$ invariant, i.e. $\exp(i\lambda x)\mathscr{D} \subset \mathscr{D}$ for $\lambda \in R$. We say that a closed operator L is of order $\leqq n$ with respect to x if for $u \in \mathscr{D}$, $\lambda \in R$,

$$L(\lambda x)u = \exp(-i\lambda x)L\exp(i\lambda x)u$$

is a polynomial in λ of degree at most n. We say that the order of L with respect to x is n and write $\operatorname{ord}_x L = n$ if L is of order $\leqq n$ but not of order $<n$.

If we introduce the derivation $D_x L = i[L, x]$, then we have

$$L(\lambda x) = \sum \lambda^k L_k/k!, \qquad L_k = D_x^k L.$$

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Clearly L_k is a homogeneous polynomial in x. If L^+ is a formal adjoint of L, i.e. $(L^+u, v)=(u, Lv)$ for $u, v \in \mathscr{D}$, then we easily see that $(L(\lambda x))^+= L^+(\lambda x)$, so that $\operatorname{ord}_x L^+=\operatorname{ord}_x L$, and $(L_k)^+=L_k^+$.

If $f \in \mathscr{L}_1(R)$ and $\hat{f}$ is its Fourier transform, then the formal computation

$$\begin{aligned} Lf(x) &= (2\pi)^{-1/2}L\int_R \hat{f}(\lambda)\exp(i\lambda x)\,d\lambda \\ &= (2\pi)^{-1/2}\int_R \hat{f}(\lambda)\exp(i\lambda x)L(\lambda x)\,d\lambda \\ &= (2\pi)^{-1/2}\int_R \hat{f}(\lambda)\exp(i\lambda x)\left(\sum \lambda^k L_k/k!\right)\,d\lambda, \end{aligned}$$

suggests the formula

$$Lf(x) = \sum i^{-k}f^{(k)}(x)L_k/k!. \tag{1}$$

This formula is valid under various hypotheses. Perhaps the simplest is to consider both sides of (1) as operators on the set $\mathscr{D}_x$ of all v of the form $v=\varphi(x)u$, $\varphi \in C_0^\infty(R)$, $u \in \mathscr{D}$. Clearly $\mathscr{D}_x$ is a dense subset of $\mathscr{D}$. Then equation (1) holds for $f \in C^{n+1}(R)$, if $\operatorname{ord}_x L=n$.

LEMMA. *If $\operatorname{ord}_x M=m$, $f, g \in C^{m+1}(R)$ and L is a linear operator such that $LM(\lambda x)=M(\lambda x)L$ for $\lambda \in R$, and if $v \in \mathscr{D}_x$, then*

$$Mf(x)Lg(x)v = \sum i^{-k}(k!)^{-1}h_k M_k v,$$

where the operators h_k are defined by the formal series

$$f(x+\lambda)Lg(x+\lambda)v = \sum \lambda^k h_k/k!.$$

PROOF. A straightforward computation yields

$$h_k = \sum \binom{k}{s} f^{(k-s)}(x)Lg^{(s)}(x).$$

Hence we obtain on the set $\mathscr{D}_x$,

$$\begin{aligned} Mf(x)Lg(x) &= \sum i^{-r}f^{(r)}(x)M_r Lg(x)/r! \\ &= \sum i^{-r}f^{(r)}(x)LM_r g(x)/r! \\ &= \sum i^{-r-s}f^{(r)}(x)Lg^{(s)}(x)M_{r+s}/r!\,s!, \end{aligned}$$

which yields the stated result.

COROLLARY. *If $\operatorname{ord}_x L<\infty$, $\operatorname{ord}_x M<\infty$, and $LM(\lambda x)=M(\lambda x)L$ for $\lambda \in R$, then for $\alpha \in R$ we have, on $\mathscr{D}_x$,*

$$ML(i\alpha x^2/2) = \sum \alpha^k L_k(i\alpha x^2/2)M_k/k!.$$

PROOF. In the lemma take $f(x)=\exp(\alpha x^2/2)=g(x)^{-1}$, and set

$$H(\lambda) = f(x+\lambda)Lg(x+\lambda) = \sum \lambda^k h_k/k!.$$

We obtain

$$H'(\lambda) = f(x+\lambda)[\alpha(x+\lambda), L]g(x+\lambda) = \alpha[x, H(\lambda)].$$

Comparing coefficients, we infer that $h_{k+1}=\alpha[x, h_k]=i\alpha D_x h_k$, so that

$$h_k = (i\alpha)^k D_x^k h_0 = (i\alpha)^k L_k(i\alpha x^2/2).$$

The corollary follows immediately.

THEOREM. *Under the hypotheses of the corollary, we have, for* $\alpha, \beta \in R$,

$$M(-i\beta x^2/2)L(i\alpha x^2/2) = \sum (\alpha+\beta)^k L_k(i\alpha x^2/2)M_k(-i\beta x^2/2)/k!.$$

PROOF. The theorem is an immediate consequence of the identity

$$M(-i\beta x^2/2)L(i\alpha x^2/2) = \exp(-\beta x^2/2)((ML(i(\alpha+\beta)x^2/2))\exp \beta x^2/2).$$

The identity $L(\lambda x)M(\mu x)=(LM((\mu-\lambda)x))(\lambda x)$ shows that the hypothesis is equivalent to the symmetric condition that $L(\lambda x)$ and $M(\mu x)$ commute for $\lambda, \mu \in R$, which is, in turn, equivalent to the condition that L_j and M_k commute for all j and k.

We shall say that L is x-normal if $LL^+(\lambda x)=L^+(\lambda x)L$ for $\lambda \in R$. This property implies that L is formally normal, i.e. that L and L^+ commute. If L is x-normal, then for $\alpha \in R$, we have

$$(L(i\alpha x^2/2))^+ = L^+(-i\alpha x^2/2).$$

The theorem implies

$$L(i\alpha x^2/2)^+L(i\alpha x^2/2) = \sum (2\alpha)^k L_k(i\alpha x^2/2)L_k(i\alpha x^2/2)^+/k!.$$

If we apply this formula to L_r, we obtain

$$L_r(i\alpha x^2/2)^+L(i\alpha x^2/2) = \sum (2\alpha)^k L_{r+k}(i\alpha x^2/2)L_{r+k}(i\alpha x^2/2)^+/k!.$$

For $v \in \mathcal{D}_x$, let

$$P_r(\beta) = \sum \beta^k \, \|L_{r+k}(i\alpha x^2/2)^+ v\|^2/k! = \sum c_{r+k}\beta^k/k!.$$

Then P_r is a polynomial with nonnegative coefficients, $P_r'=P_{r+1}$, $P_r=P_0^{(r)}$, and $P_r(2\alpha)=\|L_r(i\alpha x^2/2)v\|^2$.

For $0<\beta, \gamma$ we have

$$\begin{aligned} P_0(\beta) \leqq P_0(\beta+\gamma) &= \sum \gamma^r P_r(\beta)/r! = \sum c_k(\beta+\gamma)^k/k! \\ &\leqq ((\beta+\gamma)/\beta)^n P_0(\beta). \end{aligned}$$

If we set $\beta=2\alpha$, $\gamma=\beta/n$, we obtain

$$P_0(2\alpha) \leqq \sum (2\alpha/n)^r P_r(2\alpha)/r! \leqq eP_0(2\alpha).$$

It is also obvious that $\beta P_r'(\beta) \leqq (n-r)P_r(\beta)$, which implies that $n!\,P_0(\beta) \geqq (n-r)!\,\beta^r P_r(\beta)$. Summarizing these results, we obtain

COROLLARY. *If L is an x-normal operator and* $\operatorname{ord}_x L=n$, *and* $v \in \mathscr{D}_x$, *then for* $\alpha \geqq 0$ *we have*

$$\|L_r(i\alpha x^2/2)v\|^2 = \sum (2\alpha)^k \|L_{r+k}(i\alpha x^2/2)^+ v\|^2/k!,$$

$$\sum_{r=1}^{n} (2\alpha/n)^r \|L_r(i\alpha x^2/2)v\|^2/r! \leqq (e-1) \|L(i\alpha x^2/2)v\|^2,$$

and

$$(2\alpha)^r(n-r)!\,\|L_r(i\alpha x^2/2)v\|^2 \leqq n!\,\|L(i\alpha x^2/2)v\|^2.$$

More generally, if $\mathfrak{A}$ is any linear space of commuting selfadjoint operators, we may define the order of L with respect to $\mathfrak{A}$ by

$$\operatorname{ord}(L, \mathfrak{A}) = \max_{x\in\mathfrak{A}} \operatorname{ord}_x L.$$

We say that L is $\mathfrak{A}$-normal if it is x-normal for each $x \in \mathfrak{A}$. If $x_1, \cdots, x_N \in \mathfrak{A}$ and $Q=\sum \alpha_j x_j^2/2$, then we may extend the above results to the operator $L(iQ)$.

We need only interpret r and k as multi-indices, e.g. $k=(k_1, \cdots, k_N)$, where k_j are nonnegative integers, and

$$L_k = D^k L, \qquad D^k = \prod D_j^{k_j}, \qquad D_j L = i[L, x_j].$$

Of course α^k is the monomial $\prod \alpha_j^{k_j}$. We simply observe that the proof is valid with this reinterpretation of the notation. We may replace $\mathscr{D}_x$ by the set $\mathscr{D}'$ of all v of the form $v=\varphi(x_1, \cdots, x_N)u$, where $\varphi \in C_0^\infty(R^N)$ and $u \in \mathscr{D}$.

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