PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 1, March 1973

ON FREE ABELIAN l-GROUPS[1]

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ABSTRACT. Let F denote the free abelian lattice ordered group over an unordered torsion free group G. Necessary and sufficient conditions are given on G in order for F to be an l-subgroup of a cardinal product of integers. The result encompasses Weinberg's theorem that the freeness of G is sufficient. The corresponding embedding theorem for F is also established whenever G is completely decomposable and homogeneous.

E. Weinberg proved in [4] that a free abelian l-group is l-isomorphic to an l-subgroup of a cardinal product of integers. A simpler and more direct proof of this important result was given by S. Bernau in [1]. In this note we simplify the proof further by an application of Zorn's lemma, and at the same time we improve the theorem itself. We remark that P. Conrad [2] has given necessary and sufficient conditions for an l-group to be an l-subgroup of a cardinal product of integers, but it is not easy to determine directly whether or not a free l-group satisfies these conditions.

A set of elements $x_1, x_2, \cdots, x_n$ in an additively written abelian group is called *positively independent* if $\sum_{i=1}^{n} n_i x_i = 0$, where $n_i \geqq 0$, implies that $n_i = 0$ for each i. We emphasize that a set can be positively independent without being independent, but the terminology (unfortunately) is well established. All that we do depends basically on the following theorem or on its proof.

THEOREM 1. *Let G be any torsion free abelian group having rank at least two. If $x_1, x_2, \cdots, x_n$ are positively independent in G, there exists a nonzero pure subgroup H of G such that $x_1+H, x_2+H, \cdots, x_n+H$ are positively independent in G/H.*

PROOF. Let $\{S\}_*$ denote the pure subgroup of G generated by a subset S of G, that is, let $\{S\}_*$ denote the smallest subgroup of G such that $S \subseteq \{S\}_*$ and $G/\{S\}_*$ is again torsion free. Our notation is in agreement with [3], and it should be observed that an element x in G belongs to

Received by the editors September 1, 1971.

AMS (MOS) subject classifications (1970). Primary 06A60; Secondary 20K20.

Key words and phrases. Lattice ordered group, free group, free abelian l-group, cardinal product, subdirect product of integers, homogeneous groups.

[1] This research was supported by National Science Foundation grant GP-12318.

$\{S\}_*$ if and only if, for some positive integer n, the equation $nx=\sum n_i s_i$, where $s_i \in S$ and $n_i \in Z$, holds.

If $\{x_1, x_2, \cdots, x_n\}_* \neq G$, the theorem is trivial because we can take $H=\{h\}_*$ where h is independent of the elements $x_1, x_2, \cdots, x_n$. In particular, we may assume that $n \geqq 2$ since the rank of G exceeds 1. We proceed by induction on n. If the elements $x_1, x_2, \cdots, x_n$ are independent, then $H=\{h\}_*$, where $h=x_1+x_2+\cdots+x_{n-1}-x_n$, satisfies the theorem. Thus we may assume that the x_i's are dependent although they are positively independent. Thus we can arrange the x_i's so that, for some $k<n$,

$$\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=k+1}^{n} \alpha_i x_i \tag{1}$$

with $\alpha_n>0$ and $\alpha_i \geqq 0$ for all $i \leqq n$. Choose an arrangement of the x_i's that gives a minimal positive integer k in the dependence relation (1). If $k=1$, there exists by the induction hypothesis a nonzero pure subgroup H of G such that $x_2+H, x_3+H, \cdots, x_n+H$ are positively independent in G/H, but in view of the relation (1) with $k=1$, it quickly follows that x_1+H, $x_2+H, \cdots, x_n+H$ are also positively independent. Therefore, we may assume that $k \geqq 2$. Define $y_i=x_i$ if $i<n$ and define $y_n=-\sum_{i=2}^{k} \alpha_i x_i+\alpha_n x_n$. Assuming that the y_i's are not positively independent, we have $\sum \beta_i y_i=0$ where $\beta_i \geqq 0$ and $\beta_n>0$. Hence $\beta_n \alpha_n>0$, and $\beta_1 x_1+(\beta_2-\beta_n\alpha_2)x_2+\cdots+(\beta_k-\beta_n\alpha_k)x_k+\beta_{k+1}x_{k+1}+\cdots+\beta_{n-1}x_{n-1}+\beta_n\alpha_n x_n=0$, which leads to a contradiction on the minimality of k since there are less than k negative coefficients involved. Therefore, the y_i's are positively independent. Due to the relations $y_i=x_i$ if $i<n$,

$$y_n = -\sum_{i=2}^{k} \alpha_i x_i + \alpha_n x_n = \alpha_1 x_1 - \sum_{i=k+1}^{n-1} \alpha_i x_i$$

and the induction hypothesis, there exists a nonzero pure subgroup H of G such that $y_1+H, y_2+H, \cdots, y_n+H$ are positively independent in G/H. It is trivial to verify that $x_1+H, x_2+H, \cdots, x_n+H$ must consequently be positively independent, and the theorem is proved.

As usual, Q denotes the additive group of rational numbers.

Corollary 1. *If $x_1, x_2, \cdots, x_n$ are positively independent elements in a torsion free abelian group G, there exists a subgroup H of G such that G/H is a subgroup of Q and $x_1+H, x_2+H, \cdots, x_n+H$ are positively independent in G/H.*

Proof. By Zorn's lemma, there exists a subgroup H of G maximal with respect to being pure and having the property that $x_1+H, x_2+H, \cdots$, x_n+H are positively independent in G/H. By Theorem 1, the maximality of H implies that G/H has rank 1. Hence G/H is a subgroup of Q since it is torsion free and has rank 1.

The following corollary generalizes results in [**1**] on rational vector spaces and free abelian groups.

COROLLARY 2. *Let G be any torsion free abelian group and let $x_1, x_2, \cdots, x_n$ be positively independent elements of G. There exists a homomorphism η from G into Q such that $\eta(x_i)>0$ for each i.*

PROOF. In accordance with Corollary 1, we can choose $H\subseteq G$ so that $G/H\subseteq Q$ and $x_1+H, x_2+H, \cdots, x_n+H$ are positively independent in G/H. Let $\eta:G\to G/H$ be the natural homomorphism. Then $\eta(x_i)=x_i+H$ must be positive for all i if $\eta(x_1)>0$, for otherwise $x_1+H, x_2+H, \cdots, x_n+H$ would not be positively independent in $G/H\subseteq Q$. However, we may assume that $\eta(x_1)=1$ since any nonzero element of Q goes to 1 under some automorphism.

Before proving our main theorem, we need a lemma.

LEMMA 1. *Let $G\subseteq\prod_{\mu\in M}\{g_\mu\}$ be a subgroup of a product of infinite cyclic groups and let $x_1, x_2, \cdots, x_n$ be positively independent in G. There exists a finite subset N of M such that $x_1+H, x_2+H, \cdots, x_n+H$ are positively independent in G/H where $H=G\cap\prod_{\mu\in M-N}\{g_\mu\}$.*

PROOF. The proof of the corresponding result for independence rather than positive independence is accomplished by induction on n as follows. If $N\subseteq M$, let $\rho_N(x)$ denote the natural projection of $x\in G$ into $\prod_{\mu\in N}\{g_\mu\}$. By the induction hypothesis, we may assume that $\rho_N(x_1), \rho_N(x_2), \cdots, \rho_N(x_{n-1})$ are independent for some finite subset N of M. If the addition of $\rho_N(x_n)$ destroys the independence, write

$$\alpha_n\rho_N(x_n)=\sum_{i=1}^{n-1}\alpha_i\rho_N(x_i), \tag{2}$$

$\alpha_i\in Z$ and $\alpha_n>0$. Choose α_n minimal. Since $x_1, x_2, \cdots, x_n$ are independent, there exists $\mu\in M$ such that (2) fails to hold if $N+=\{N,\mu\}$ replaces N. We claim that $\rho_{N+}(x_1), \rho_{N+}(x_2), \cdots, \rho_{N+}(x_n)$ are independent. Otherwise, we have

$$\beta_n\rho_{N+}(x_n)=\sum_{i=1}^{n-1}\beta_i\rho_{N+}(x_i)$$

with $\beta_n>0$, which implies in particular that

$$\beta_n\rho_N(x_n)=\sum_{i=1}^{n-1}\beta_i\rho_N(x_i).$$

Since α_n was chosen minimal in (2), $\beta_n=\alpha_n q$ for some positive integer q. This implies, in turn, that $\beta_i=\alpha_i q$ for $i\leqq n$ since $\rho_N(x_1)$, $\rho_N(x_2), \cdots$, $\rho_N(x_{n-1})$ are independent. This, however, yields

$$\alpha_n\rho_{N+}(x_n)=\sum_{i=1}^{n-1}\alpha_i\rho_{N+}(x_i)$$

and a contradiction on the choice of μ and $N+=\{N,\mu\}$. Hence we may assume that $x_1, x_2, \cdots, x_n$ are dependent although positively independent. Write

$$\sum_{i=1}^{k}\alpha_i x_i=\sum_{i=k+1}^{n}\alpha_i x_i$$

with $\alpha_n>0$ and $\alpha_i\geqq 0$ for each i, and proceed exactly the same as in the proof of Theorem 1.

THEOREM 2. *Let G be any torsion free abelian group with the trivial partial order; the only positive element of G is zero. Let F be the free abelian l-group over G. Then F is an l-subgroup of a cardinal product of integers if and only if G (as an unordered group) is a subgroup of a product of integers.*

PROOF. Since G is a subgroup of F, the fact that F is an l-subgroup of a cardinal product of integers implies *a priori* that G is a subgroup of a product of integers. Conversely, suppose that G is a subgroup of a product of integers. In order to show that F can be embedded as an l-subgroup in a (strong) cardinal product of integers, it suffices to show that the l-homomorphisms from F to Z separate the points of F. Further, the separation of the positive elements is, of course, enough, so all that we need to finish the proof of the theorem is to show that if $x>0$ in F then there exists an l-homomorphism ϕ from F to Z such that $\phi(x)>0$.

Let x be an arbitrary strictly positive element of F. Write, in the familiar fashion,

$$x=\bigvee_i\bigwedge_j\langle x_{i,j}\rangle, \tag{3}$$

where the join and meet are computed in the cardinal product $\prod_{t\in T}\boxplus G_t$ with $\{G_t\}_{t\in T}$ denoting the collection of all possible total orders on the group G. In other words, G_t is G endowed with a total order t, and each total order of G is represented exactly once. If $g\in G$, $\langle g\rangle$ denotes the element in $\prod_{t\in T}\boxplus G_t$ that has g for each t-component. Thus in the representation (3) we interpret $x_{i,j}$ to be an element of G. The condition $x>0$ in $F\subseteq\prod_{t\in T}\boxplus G_t$ implies that x_t, the t-component of x, is positive for each t and is strictly positive for some t. Hence, for some t, $\bigvee_i\bigwedge_j x_{i,j}>0$ in G_t, which implies that $\bigwedge_j x_{i,j}>0$ in G_t for some i since G_t is totally ordered.

Therefore, for some i, $x_{i,1}, x_{i,2}, \cdots, x_{i,j(i)}$ are positively independent in G with $j(i)$ denoting the range on j for the fixed subscript i.

Let $G \subseteq \prod_{\mu\in M} \{g_\mu\}$. Since $x_{i,1}, x_{i,2}, \cdots, x_{i,j(i)}$ are positively independent, there exists by Lemma 1 a finite subset N of M such that $x_{i,1}+H$, $x_{i,2}+H, \cdots, x_{i,j(i)}+H$ are positively independent in G/H, where $H = G \cap \prod_{\mu\in M-N} \{g_\mu\}$. By Corollary 2, we know that there exists a homomorphism η from G/H into Q such that $\eta(x_{i,j})>0$ for each $j \leqq j(i)$. However, $G/H \subseteq \prod_{\mu\in N} \{g_\mu\}$ is finitely generated since N is finite, and therefore, $\eta(G/H)$ is cyclic because Q is locally cyclic. We conclude that there is a homomorphism $\pi: G \to G/H \to Z$ from G into Z that maps $x_{i,j}$ onto a positive integer; it is understood that i is still appropriately fixed and that $j \leqq j(i)$. From the commutative diagram

$$\begin{tikzcd} & F \arrow[dr, "\phi"] & \\ G \arrow[ur] \arrow[rr, "\pi"] & & Z, \end{tikzcd}$$

where ϕ is an l-homomorphism, we see that

$$\phi(x) = \phi\left(\bigvee_i \bigwedge_j \langle x_{i,j}\rangle\right) = \bigvee_i \bigwedge_j \phi(\langle x_{i,j}\rangle) = \bigvee_i \bigwedge_j \pi(x_{i,j}) > 0.$$

This shows that the l-homomorphisms from F to Z separate the points of F, and the theorem is proved.

It is customary to call the free abelian l-group over an unordered free abelian group G of rank m simply the free abelian l-group of rank m. We shall denote the free abelian l-group of rank m by F_m. If $G = \sum_m Z$ is free, then $G \subseteq \prod_m Z$, so Theorem 2 contains in particular the following result.

COROLLARY 3 (WEINBERG). *For any cardinal m, the free abelian l-group F_m is an l-subgroup of a cardinal product of integers; equivalently, F_m is a subdirect (cardinal) product of integers since any nonzero subgroup of Z is order isomorphic to Z.*

The next corollary is a direct generalization of Corollary 3.

COROLLARY 4. *Let G be any completely decomposable, homogeneous, torsion free group of type τ. Let F be the free abelian l-group over G. Then F is an l-subgroup of a cardinal product of copies of the rank 1 group of type τ.*

PROOF. Let A be the (unique) torsion free group of rank 1 and type τ. Under the hypothesis on G, we have $G = \sum_M A$ for some cardinal M.

Let $x_1, x_2, \cdots, x_n$ be positively independent in G. By Lemma 1, there exists a positive integer N such that $x_1+H, x_2+H, \cdots, x_n+H$ are positively independent in G/H where $G/H=\sum_N A$ (with $H=\sum_{c(N)} A$ where $c(N)$ denotes the complement of N in M). Therefore, there exists a homomorphism ϕ from G/H into Q such that $\phi(x_i+H)>0$ for each $i\leqq n$. By [**3**, Theorem 46.8], $\phi(G/H)$ is (isomorphic to) a direct summand of G/H since the kernel of ϕ is pure due to the fact that the image of ϕ is torsion free. Consequently, $\phi(G/H)=A$, and we conclude, as in the proof of Theorem 2, that the l-homomorphisms from F to A separate the points of F.

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