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# THE CRITICAL POINTS OF A TYPICALLY-REAL FUNCTION

A. W. GOODMAN[1]

ABSTRACT. The critical points of a typically-real function cannot lie too close to the real axis. By adding a mild restriction, we determine $D_k$ the domain of variability of a $k$th order critical point. Similar results are obtained for a $k$th order branch point. We determine the domain of univalence for typically-real functions and propose a reasonable conjecture for the domain of $k$-valence.

1. **Introduction.** A function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

regular in $E:|z|<1$, is said to be typically-real if it satisfies the condition

$$(\Im f(z))(\Im z) > 0 \tag{2}$$

for all nonreal $z$ in $E$. This class of functions (which we denote by TR) was introduced by Rogosinski [9] in 1932 and has been the object of many investigations ([2], [3], [7]).

The condition (2) implies that $f(z)$ is real in $E$, if and only if $z$ is real. Further it implies that if $-1<z<1$, then $f'(z)>0$. It is intuitively obvious that if $c$ is a critical point of $f(z)$, a point where $f'(z)=0$, then $c$ cannot lie too close to the real axis. In this work we determine this forbidden domain precisely. More generally, if $c_k$ is a critical point of $k$th order, it cannot lie too close to the real axis. In this case we advance the conjecture that a certain domain $D_k$ is the forbidden domain. If we add a suitable condition on $f(z)$, then we can prove that $c_k \notin D_k$. We also obtain similar results about the location of a branch point $b_k=f(c_k)$ when $f(z)\in$ TR. The work closes with a theorem on the domain of univalence of the class TR and a conjecture on the domain of $k$-valence of the class TR. Our main tool is the theory of subordination, but we also use a Stieltjes integral representation due to M. S. Robertson [7].

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2. **The example function.** In any subordination argument we need an example function which is a superordinate function for the problem under consideration. Let $\hat{S}_k$ be the surface obtained by attaching to a base plane $k$ half planes $\Im w>0$ at a branch point $B_k=|B_k|i$; and $k$ half planes $\Im w<0$ at $\bar{B}_k$, where $B_k$ is yet to be determined. Let $F_k(z)$ be the function in the class TR that maps $E$ onto the surface $\hat{S}_k$. We will obtain $F_k(z)$ explicitly by a sequence of transformations. We consider first the half disk $E^{(l)}\equiv E\cap\{z|\Re z<0\}$. The linear transformation $u=-(1+zi)/(i+z)$ maps $E^{(l)}$ onto the first quadrant; then $v=u^2$ carries this quadrant onto the upper half plane, and finally $w=-i(v-i)/(v+i)$ takes this half plane onto $E$. The composition of these mappings gives

$$T_1(z)\equiv\frac{1+2z-z^2}{1-2z-z^2}, \tag{3}$$

a function that carries $E^{(l)}$ onto $E$. It is important to note that: $T_1(0)=1$, $T_1(i)=i$, $T_1(-1)=-1$, and $T_1(-i)=-i$. Hence the diameter from $-i$ to $i$ is mapped onto the arc $-i$, 1, $i$ of the boundary of $E$.

Next, with $k$ a fixed natural number, let $\eta=e^{\pi i/(k+1)}$, and let

$$T_2(z)=(z+s)/(1+sz), \tag{4}$$

where $s$ is real and adjusted so that $T_2$ maps the arc $-i$, 1, $i$ onto the arc $\bar{\eta}$, 1, $\eta$. A brief computation gives

$$s=\frac{\cos(\pi/(k+1))}{1+\sin(\pi/(k+1))}. \tag{5}$$

The function

$$T_3(z)=(1-z^{k+1})/(1+z^{k+1}) \tag{6}$$

carries $E$ onto a domain consisting of $k+1$ half planes $\Re w>0$. It is worth noting that $T_3$ maps the arc $\bar{\eta}$, 1, $\eta$ onto the imaginary axis.

Finally we define $F_k(z)$ by

$$F_k(z)\equiv\frac{i}{2(k+1)A}T_3(T_2(T_1(iz))) \tag{7}$$

where

$$A=(1-s)/(1+s). \tag{8}$$

By our construction $F_k(z)$ maps the half disk $E^{(+)}\equiv E\cap\{z|\Im z>0\}$ onto the surface formed by $k+1$ half planes $\Im w>0$, tied together with a $k$th order branch point. If we reflect the half disk across the real axis, and the image domain across the real axis, we find that $F_k(z)$ maps $E$ onto $\hat{S}_k$ as required.

A computation shows that

$$F_k(z) = \frac{i}{2(k+1)A}\frac{P^{k+1}(z) - Q^{k+1}(z)}{P^{k+1}(z) + Q^{k+1}(z)}, \tag{9}$$

where

$$P(z) \equiv 1 - 2Aiz + z^2, \qquad Q(z) \equiv 1 + 2Aiz + z^2. \tag{10}$$

The factor $1/2(k+1)A$ is selected so that $F_k'(0)=1$. We have

LEMMA 1. *For each positive integer $k$, the function $F_k(z)$, defined by equations* (5), (8), (9), *and* (10), *maps $E$ onto the surface $\hat{S}_k$. Further $F_k(z) \in \mathrm{TR}$. $F_k(z)$ has two critical points of $k$th order, one at*

$$C_k \equiv R_k i \equiv ((A^2+1)^{1/2} - A)i, \tag{11}$$

*and the other at $-C_k$. The corresponding branch points are at $\pm i/2(k+1)A \equiv \pm B_k$.*

*In particular, $k=1, 2$ gives*

$$F_1(z) = \frac{z(1+z^2)}{(1-z^2)^2}, \qquad R_1 = \sqrt{2} - 1, \qquad B_1 = i/4, \tag{12}$$

*and*

$$F_2(z) = \frac{z(9+14z^2+9z^4)}{9(1-z^2)^2(1+z^2)}, \qquad R_2 = \sqrt{3}/3, \qquad B_2 = i/2\sqrt{3}. \tag{13}$$

3. **The critical points.** Let $D_k$ be the domain that is bounded above by the arc of the circle through the points $-1$, $R_k i$, and 1, and bounded below by the arc of the circle through $-1$, $-R_k i$ and 1. Here $R_k$ is defined by equations (5), (8) and (11). Then we have

THEOREM 1. *Suppose that $f(z) \in \mathrm{TR}$ and that $f(z)$ has a $k$th order critical point at $c_k$. Suppose further that $f(c_k)=b_k$, and that the equation $f(z)=b_k$ has no solution in $E$, except $z=c_k$. Then the point $c_k$ must lie in $E-D_k$. Further, for each $c_k$ in $E-D_k$ there is an $f(z) \in \mathrm{TR}$ that has a $k$th order critical point at $c_k$. If $c_k$ is on the boundary of $D_k$ and $c_k \neq \pm 1$, then $f(z)$ is unique.*

PROOF. Since each function in TR has real coefficients, the image of $E$ is symmetric with respect to the real axis. Hence we can restrict our attention to critical points that lie in $E^{(+)}$ and branch points that lie in the upper half plane. If $f(z) \in \mathrm{TR}$ and $r$ is in $(-1, 1)$, then

$$\phi(z) \equiv \left(f\left(\frac{z+r}{1+rz}\right) - f(r)\right) \Big/ (1-r^2)f'(r) \tag{14}$$

is also in TR. Since $(z+r)/(1+rz)$ moves the critical point along an arc of a circle through the points $-1$ and 1, we can select $r$ so that $\phi(z)$ has its corresponding critical point on the imaginary axis. Hence without loss of generality we may assume that $\Re c_k=0$ and $\Im c_k>0$. Consequently it will be sufficient to prove that $|c_k|\geqq|C_k|\equiv R_k$. Next, we observe that if $f(z)\in$ TR, then $-f(-z)$ is also in TR. Hence if $b_k=\alpha+i\beta$ is the corresponding branch point we may assume that $\alpha\geqq 0$, and $\beta>0$.

We now adjust our example function $F_k(z)$ so that it has the same branch point. Since $F_k(z)$ maps $(-1, 1)$ onto the real axis there is a real $t$ such that $F_k(t)=\alpha|B_k|/\beta$. Further, we set $a=\beta/|B_k|>0$. Then a brief computation shows that

$$H(z) \equiv a\left[F_k\left(\frac{z-t}{1-tz}\right) + F_k(t)\right] \tag{15}$$

has a $k$th order critical point at $C_k^\star=(C_k+t)/(1+tC_k)$ and that $H(C_k^\star)=\alpha+i\beta=b_k$. Since $F_k(z)$ is an odd function, we also have $H(0)=0$. If $H(E)=S^\star$, then $S^\star$ is merely a translation followed by an expansion (or a contraction) of $\hat{S}$.

Let $z=G(w)$ be the inverse function of $H(z)$ defined on $S^\star$. Let us assume for the moment that the composite function $J(z)\equiv G(f(z))$ is well-defined. If so, it satisfies the conditions of Schwarz's lemma because $f(z)$ maps $E$ onto a surface $S$ that is "carried" by $S^\star$, and $G$ takes $S^\star$ onto $E$. Further, $J(0)=G(f(0))=G(0)=0$. Consequently for any $z$ in $E$, $|J(z)|\leqq|z|$, and because of our normalization the equality sign holds if and only if $J(z)\equiv z$. Now $J(c_k)=G(f(c_k))=G(B_k)=(C_k+t)/(1+tC_k)$. Hence

$$|c_k| \geqq |(R_ki + t)/(1 + tR_ki)| \geqq R_k,$$

with equality only if $t=0$.

It remains to show that $J(z)$ is well-defined. This is the case if $f(z)$ is subordinate to $H(z)$. The concept of subordination was first used by Lindelöf [**4**], but the terminology was suggested by Littlewood [**5**]. The method was extensively exploited by Littlewood [**5**], Nehari [**6**], Rogosinski ([**8**], [**10**], [**11**], [**12**]), and others [**2**]. In most of the applications the superordinate function is either univalent or locally univalent. Whenever critical points occur in $H(z)$, the problem is avoided by adopting an alternate definition: $f(z)$ is subordinate to $H(z)$ if there is a $J(z)$ such that $f(z)=H(J(z))$ and $J(z)$ satisfies the conditions of Schwarz's lemma. Littlewood [**5**], and Beckenbach and Graham [**1**] give the following geometric criterion for $f(z)$ to be subordinate to $H(z)$.

Let $f(E)=S$ and let $H(E)=S^\star$. If there is a mapping of $S$ into $S^\star$ such that each point $w$ of $S$ goes into a point of $S^\star$ with the same complex coordinate, and each closed contour on $S$ beginning and ending at $f(0)$

goes into a closed contour of $S^\star$ beginning and ending at $H(0)$, then $f(z)$ is subordinate to $H(z)$, and the representation $f(z)=H(J(z))$ is indeed possible.

A mapping of the type just described will be called an $S^\star$-projection. In our problem $f(0)=H(0)=0$ and both functions have no other zeros in $E$. Both functions are univalent in a neighborhood of $z=0$, and hence there is an $S^\star$-projection in a neighborhood of $w=0$. Let $\gamma_0$ be the radial segment from $z=0$ to $z=c_k$ and let $\Gamma_0=f(\gamma_0)$. Since the inverse of $H(z)$ is locally univalent except for the point $w=b_k=B_k$, there is an $S^\star$-projection of $\Gamma_0$ onto a curve $\Gamma_0^\star$ of $S^\star$.

If $\gamma$ is any radial segment that does not pass through $z=c_k$ or $z=\bar{c}_k$ and $\Gamma=f(\gamma)$, then the $S^\star$-projection of $\Gamma$ established near $w=0$ can be continued over all of $\Gamma$ because $f(z)\neq b_k$, $\bar{b}_k$ on $\Gamma$, and $S^\star$ is locally univalent everywhere except at $b_k$ and $\bar{b}_k$. Finally consider the points $w=f(z)$ where $z$ is on the extension of the radial segment to $c_k$. Since every simple closed curve that encloses $c_k$ (and not $\bar{c}_k$) has an image under $f(z)$ that winds around $b_k$, $k$ times, and the same is true of the image under $H(z)$, the $S^\star$-projection can be extended to the remaining points of $S$. It is clear that the correspondence just described satisfies the Littlewood requirements, and hence $f(z)$ is subordinate to $H(z)$.

To establish an $S^\star$-projection it was necessary to assume that $f(z)=b_k$ has no solution in $E$ other than $z=c_k$. We conjecture that Theorem 1 is true without this hypothesis. A proof for the case $k=1$ will be given in §5.

4. **The branch points.** Let $F_k(z)$ be the extremal function defined in §2, and for each real $t$ in $(-1, 1)$, set

$$F_k(z, t) = \left(F_k\left(\frac{z-t}{1-tz}\right) + F_k(t)\right)\Big/F_k'(t)(1-t^2). \tag{16}$$

The function $F_k(z, t)$ has one critical point on the upper boundary of $D_k$, and as $t$ varies, this critical point (together with its conjugate) describes the boundary of $D_k$ (except for the points $\pm 1$). The corresponding branch points describe two curves, $\Gamma_k^{(+)}(w)$ in the upper half plane, and $\Gamma_k^{(-)}(w)$ in the lower half plane. Since $F_k(C_k)=i/2(k+1)A\equiv B_k$ the curve $\Gamma_k^{(+)}(w)$ has the parametric representation $w=(F_k(t)+|B_k|i)/(F_k'(t)(1-t^2))$. A brief computation will show that for each fixed $k$, the curve $\Gamma_k^{(+)}(w)$ is starlike with respect to the origin and that as $t\to\pm 1$, $w\to\pm\frac{1}{4}$.

Let $D_k(w)$ be the union of the real axis and the domain bounded above by $\Gamma_k^{(+)}(w)$ and below by $\Gamma_k^{(-)}(w)$. With this notation we have

THEOREM 2. *Let $f(z)$ satisfy the conditions of Theorem 1. Then $b_k\notin D_k(w)$. For each point $b_k$ in the complement of $D_k(w)$, there is an $f(z)\in$ TR with a* *kth order branch point at $b_k$. If $b_k$ is on $\Gamma_k^{(+)}(w) \cup \Gamma_k^{(-)}(w)$ then there is only one such function.*

PROOF. As in §3, we select $a$ and $t$ so that $f(z)$ and $H(z)$ have the same branch point at $b_k$. If $G(w)$ is the inverse function of $H(z)$ on $S^\star$, then $J(z) \equiv G(f(z))$ satisfies the conditions of Schwarz's lemma. Consequently $0 < J'(0) \leqq 1$, and equality occurs if and only if $J(z) \equiv z$. But

$$J'(0) = \frac{f'(0)}{aF_k'(-t)(1-t^2)} = \frac{1}{aF_k'(t)(1-t^2)} \leqq 1. \tag{17}$$

Hence $a > 1/F_k'(t)(1-t^2)$ and consequently $H(z) = \mu F_k(z, t)$ where $\mu \geqq 1$, with equality if and only if $f(z) = F_k(z, t)$. Therefore the branch point of $f(z)$ lies at the end point of a radial segment that terminates on or passes through $\Gamma_k^{(+)}(w) \cup \Gamma_k^{(-)}(w)$. Since each of these curves is starlike with respect to the origin, this completes the proof.

5. **A remark on valence.** W. E. Kirwan [3] has proved that the radius of univalence for the class TR is $\sqrt{2}-1$. Since $C_1 = (\sqrt{2}-1)i$ and since the transformation (14) moves this critical point along the upper boundary of $C_1$, Kirwan's result will give Theorem 1 when $k=1$.

I am indebted to E. B. Saff for calling my attention to the paper by Kirwan. Saff also suggested that perhaps Kirwan's circle of univalence could be enlarged to include the domain $D_1$.

THEOREM 3. *Let $f(z) \in$ TR. Then $f(z)$ is univalent in $D_1$ and $D_1$ is the maximal domain of univalence for the class* TR.

PROOF. By the symmetry we can restrict ourselves to the upper half disk $E^{(+)}$. Let $f(z_1) = f(z_2)$ with $z_1$ and $z_2$ in $E^{(+)}$. There is a minimal arc of a circle through $\pm 1$ on which univalence fails, and hence we can assume that $z_1$ and $z_2$ lie on this arc. By a transformation of the type (14) we may further assume that $z_1$ and $z_2$ are symmetric with respect to the $y$-axis. Let $z_2 = x+iy$, $z_1 = -x+iy$ where $x>0$ and $y>0$.

M. S. Robertson [7] proved that each function in TR has a Stieltjes integral representation

$$f(z) = \frac{1}{\pi}\int_0^\pi \frac{z\, d\mu(\theta)}{1 - 2z\cos\theta + z^2}, \qquad \int_0^\pi d\mu(\theta) = \pi, \tag{18}$$

where $\mu(\theta)$ is nondecreasing on $[0, \pi]$. Consequently if $f(z_1) = f(z_2)$ we have

$$
(19)\qquad
\begin{aligned}
0 &= \int_0^\pi \left(\frac{z_2}{1 - 2z_2\cos\theta + z_2^2} - \frac{z_1}{1 - 2z_1\cos\theta + z_1^2}\right) d\mu(\theta) \\
&= \int_0^\pi \frac{(z_2 - z_1)(1 - z_1z_2)\, d\mu(\theta)}{\text{Denominator}}.
\end{aligned}
$$

Since $(z_2-z_1)(1-z_1z_2)\neq 0$ we can write that

$$
(20)\qquad 0 = \int_0^\pi \frac{d\mu(\theta)}{\text{Den.}}.
$$

Now equation (20) is impossible if $\Re(\text{Den.})>0$. But

$$
\begin{aligned}
\Re(\text{Den.}) &= 1 - 2x^2 - 6y^2 + (x^2 + y^2)^2 + 4(x^2 + y^2)\sin^2\theta \\
&\geqq 1 - 2x^2 - 6y^2 + (x^2 + y^2)^2.
\end{aligned}
$$

$$
(21)\qquad \Re(\text{Den.}) \geqq (1 - x^2 - y^2 - 2y)(1 - x^2 - y^2 + 2y),
$$

with equality only if $\theta=0$, or $\pi$ ($\mu(\theta)$ has jumps only at 0 or $\pi$). Since the two factors on the right side of (21) will give equations for the circles that form the boundary of $D_1$ we see that $\Re(\text{Den.})>0$ in $D_1$.

It is clear that if $f(z)\in \mathrm{TR}$, the image of $D_1$ under $f(z)$ need not be star-like with respect to $w=0$. However, it seems likely that the image will be convex in the direction of the imaginary axis.

It also seems likely that every $f(z)\in\mathrm{TR}$ is at most $k$-valent in $D_k$ for every natural number $k$. If so, then our example functions show that $D_k$ is the maximal domain of $k$-valence for the class TR.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620