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A NEW SIMPLE LIE ALGEBRA OF CHARACTERISTIC THREE

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ABSTRACT. We define a restricted simple algebra T of dimension 18 over an arbitrary field of characteristic 3. From a certain property of its Cartan decomposition, we show T to be nonisomorphic to any known algebra of identical dimension.

0. The algebra T furnishes the first instance of a graded simple Lie algebra:

$$L = L_{-1} \oplus L_0 \oplus \cdots \oplus L_n, \qquad [L_i, L_j] \subseteq L_{i+j}, \tag{0.1}$$

in which L_0 is a solvable algebra of dimension greater than 1.

Contained in T is a 10-dimensional simple restricted graded algebra S, with $S_i \subseteq T_i$, and S_0 solvable, whose newness is still an open question.[1]

1. **Definition of T.** Let F be a field of characteristic 3. The algebras S and T, alluded to above, are realized as subalgebras of the Witt-Jacobson algebra W_3 over F. This algebra is spanned by derivations:[2]

$$A = (a_1, a_2, a_3) = a_1\Delta_1 + a_2\Delta_2 + a_3\Delta_3,$$

where $a_i \in F[x_1, x_2, x_3]$ with $x_i^3=0$, and Δ_i denotes the differential operator $\partial/\partial x_i$. If $B=(b_1, b_2, b_3)$, multiplication in W_3 is given by $[A, B]=C=(c_1, c_2, c_3)$, where

$$c_i = \sum_j [(\Delta_j a_i)b_j - (\Delta_j b_i)a_j]. \tag{1.1}$$

The two algebras have nested gradations

$$\begin{gathered} S = S_{-1} \oplus S_0 \oplus S_1, \\ T = T_{-1} \oplus T_0 \oplus T_1 \oplus T_2 \oplus T_3, \\ [S_i, S_j] \subseteq S_{i+j}, \quad [T_i, T_j] \subseteq T_{i+j}, \qquad S_i \subseteq T_i, \end{gathered} \tag{1.2}$$

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[1] Although R. Wilson has shown S to be nonisomorphic to the classical matrix algebra of type B_2, the possibility still remains that S is one of the 10-dimensional algebras of **[1]**, **[5]**, or **[6]**.

[2] Cf. **[4]**.

where the subspaces S_i and T_i have the following bases over F:

$$
\begin{aligned}
T_{-1} = S_{-1} &= \langle \Delta_1, \Delta_2, \Delta_3 \rangle, \\
S_0 &= \langle A_1 = (x_1, x_2, x_3),\ A_2 = (0, x_2, -x_3), \\
&\qquad A_3 = (x_2, x_3, 0),\ A_4 = (0, x_1, -x_2) \rangle, \\
S_1 &= \langle B_1 = (x_1x_2, x_1x_3, -x_2x_3),\ B_2 = (x_1^2, x_1x_2, x_2^2), \\
&\qquad B_3 = (-x_2^2, x_2x_3, x_3^2) \rangle, \\
T_0 &= S_0 \oplus \langle A_5 = (x_3, 0, 0) \rangle, \\
T_1 &= S_1 \oplus \langle B_4 = (x_1x_3, 0, x_3^2),\ B_5 = (x_2x_3, -x_3^2, 0), \\
&\qquad B_6 = (x_3^2, 0, 0) \rangle, \\
T_2 &= \langle C_1 = (x_2^2x_3 - x_1x_3^2, x_2x_3^2, 0), \\
&\qquad C_2 = (x_1^2x_3 - x_1x_2^2, x_1x_2x_3, x_2^2x_3),\ C_3 = (x_1x_2x_3, -x_1x_3^2, x_2x_3^2) \rangle, \\
T_3 &= \langle D_1 = (x_1x_2^2x_3 + x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2) \rangle.
\end{aligned}
\tag{1.3}
$$

THEOREM 1.1. *The algebras S and T are restricted central simple algebras with a natural gradation such that $S_0^{(4)} = T_0^{(4)} = 0$.*

PROOF. We verify at once that

$$
\begin{aligned}
[S_i, S_1] &= S_{i+1} \qquad (i = 0, 1), \\
[T_i, T_{-1}] &= T_{i-1} \qquad (i = 1, 2, 3), \\
[T_0, T_3] &= T_3.
\end{aligned}
\tag{1.4}
$$

The simplicity of S and T follows at once from (1.4) and the fact that the set of transformations induced in S_{-1}, T_{-1} by multiplication by elements of S_0 and T_0, respectively, is irreducible.[3] Indeed if $\mathfrak{A} \neq 0$ is an ideal of S, then for some $0 \leqq r \leqq 2$, $\mathfrak{A}(\operatorname{ad} S_{-1})^r \neq 0 \subseteq S_{-1} \cap \mathfrak{A}$, and the irreducible representation of $S_0 \to \operatorname{Hom} S_{-1}$ then implies that $\mathfrak{A} \supseteq S_{-1}$. But then, by (1.4), $\mathfrak{A} \supseteq S_0 \oplus S_1$, $\mathfrak{A} = S$, and S is central simple. Similarly if $\mathfrak{A} \neq 0$ is an ideal of T, $\mathfrak{A} \supseteq T_{-1}$ and, by (1.4), $\mathfrak{A} \supseteq$ all T_i, and T is central simple also.

The restrictedness of S and T follows at once from the restrictedness of S_0 and T_0, respectively.[4] Indeed, denoting by A^3 in W_3 the third iterate of the derivation A, it is easily verified that $A_1^3 = A_1$, $A_2^3 = A_2$, while $A^3 = 0$ for all remaining basis elements of S and T.

We finally observe that the derived algebras of S_0 and T_0 have the following bases over F:

$$
\begin{aligned}
S_0^{(2)} &= \langle A_1, A_3, A_4 \rangle, & S_0^{(3)} &= \langle A_1 \rangle, & S_0^{(4)} &= 0. \\
T_0^{(2)} &= \langle A_1, A_3, A_4, A_5 \rangle, & T_0^{(3)} &= \langle A_1, A_3 \rangle, & T_0^{(4)} &= 0.
\end{aligned}
\tag{1.5}
$$

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[3] Theorem 4.3 of [3] states that a naturally graded subalgebra G of the Witt-Jacobson algebra W_n containing all $\partial/\partial x_i$ is simple if and only if $G = G^2$, $G_0 = [G_{-1}, G_1]$, $G_1 = [G_1, G_0]$ and the representation of G_0 in G_{-1} is irreducible.

[4] Cf. Theorem 3.3 of [3].

2. **Cartan decomposition.** The subspace $H=\langle A_1, A_2\rangle$ is an abelian subalgebra of S and T. For $w \in H^*$, define

$$(2.1)\qquad \begin{aligned} T_w &= \{t \in T \mid t\,\mathrm{ad}(A) = w(A)t \text{ for all } A \in H\},\\ S_w &= \{s \in S \mid s\,\mathrm{ad}(A) = w(A)s \text{ for all } A \in H\}. \end{aligned}$$

If $w_i(A_j)=\delta_{ij}$ $(i, j=1, 2)$, it follows directly that

$$(2.2)\qquad \begin{aligned} H = T_0 &= \langle A_1, A_2\rangle, & &\\ T_{w_1} &= \langle B_2, B_5\rangle, & T_{-w_1} &= \langle \Delta_1, C_3\rangle,\\ T_{w_2} &= \langle A_3, D_1\rangle, & T_{-w_2} &= \langle A_4, A_5\rangle,\\ T_{w_1+w_2} &= \langle B_1, B_6\rangle, & T_{-w_1-w_2} &= \langle \Delta_2, C_2\rangle,\\ T_{w_1-w_2} &= \langle B_3, B_4\rangle, & T_{-w_1+w_2} &= \langle \Delta_3, C_1\rangle. \end{aligned}$$

$$(2.3)\qquad \begin{aligned} H = S_0 &= \langle A_1, A_2\rangle, & &\\ S_{w_1} &= \langle B_2\rangle, & S_{-w_1} &= \langle \Delta_1\rangle,\\ S_{w_2} &= \langle A_3\rangle, & S_{-w_2} &= \langle A_4\rangle,\\ S_{w_1+w_2} &= \langle B_1\rangle, & S_{-w_1-w_2} &= \langle \Delta_2\rangle,\\ S_{w_1-w_2} &= \langle B_3\rangle, & S_{-w_1+w_2} &= \langle \Delta_3\rangle. \end{aligned}$$

Thus H is a splitting Cartan subalgebra of both S and T, with roots $\alpha=\lambda_1 w_1+\lambda_2 w_2$ for integers $\lambda_i=-1, 0, 1$.

3. **Newness of T.** The only known simple algebra of dimension 18 is the Witt-Jabobson algebra W_2. As shown in [2], every Cartan subalgebra of W_2 is conjugate to one and only one of

$$H_1 = \langle (x_1, 0), (0, x_2)\rangle, \quad H_2 = \langle (x_1 + 1, 0), (0, x_2)\rangle,$$
$$H_3 = \langle (x_1 + 1, 0), (0, x_2 + 1)\rangle.$$

If H is a Cartan subalgebra of a Lie algebra L, let $n(L, H)$ denote the number of pairs (unordered) of roots $\{\alpha, -\alpha\}$ such that $[L_\alpha, L_{-\alpha}]=H$. Then $n(L, H)$ depends only on the conjugacy of H. We prove[5]

LEMMA 3.1. *If H is a Cartan subalgebra of W_2, then $n(W_2, H)\geqq 2$.*

PROOF. By writing $H=\langle \theta_1=(y_1, 0), \theta_2=(0, y_2)\rangle$, where $y_1=x_1$ or x_1+1, $y_2=x_2$ or x_2+1, we can prove the lemma for all three H_i at once. Let

$$U_w = \{u \in W_2 \mid u\,\mathrm{ad}(\theta) = w(\theta)u \text{ for all } \theta \in H\}.$$

[5] The author is indebted to R. Wilson for suggesting a proof based on [2] much simpler than her original one. The related proof given here is even shorter.

Letting $w_i(\theta_j)=\delta_{ij}$ for $i, j=1, 2$, we determine

$$U_{w_1} = \langle(y_1^2, 0), (0, y_1y_2)\rangle, \qquad U_{-w_1} = \langle(1, 0), (0, y_1^2y_2)\rangle,$$
$$U_{w_2} = \langle(y_1y_2, 0), (0, y_2^2)\rangle, \qquad U_{-w_2} = \langle(y_1y_2^2, 0), (0, 1)\rangle.$$

It is at once immediate that $[U_{w_1}, U_{-w_1}]=[U_{w_2}, U_{-w_2}]=H$ for all allowable substitutions for y_1 and y_2. Thus $n(W_2, H)\geqq 2$.

THEOREM 3.1. *The algebra T is not isomorphic to W_2 and is therefore new.*

PROOF. For $\alpha=w_1, w_2, w_1+w_2$ the subspace $[T_\alpha, T_{-\alpha}]$ is equal to $\langle A_1+A_2\rangle$, $\langle A_1\rangle$, $\langle A_1-A_2\rangle$, respectively. While $[T_{w_1-w_2}, T_{-w_1+w_2}]=H$. Hence $n(T, H)=1$, and by Lemma 3.1, T cannot be isomorphic to W_2.

BIBLIOGRAPHY

1. G. Brown, *Lie algebras of characteristic three with nondegenerate Killing form*, Trans. Amer. Math. Soc. **137** (1969), 259–268. MR **39** #2825.

2. S. P. Demuškin, *Cartan subalgebras of the simple Lie algebra W_n and S_n*, Sibirsk. Mat. Ž. **11** (1970), 310–325=Siberian Math. J. **11** (1970), 233–245. MR **41** #6919.

3. M. Frank, *On a theory relating matric Lie algebras of characteristic p and subalgebras of the Witt-Jacobson algebra*, Progress Report I.T., Math. Dept., University of Minnesota, Minneapolis, Minn., 1943, pp. 107–121.

4. N. Jacobson, *Classes of restricted Lie algebras of characteristic p*. II, Duke Math. J. **10** (1943), 107–121. MR **4**, 187.

5. I. Kaplansky, *Lie algebras of characteristic p*, Trans. Amer. Math. Soc. **89** (1958), 149–183. MR **20** #5799.

6. A. I. Kostrikin, *A parametric family of simple Lie algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 744–756=Math. USSR Izv. **4** (1970), 751–764. MR **43** #302.

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