REALIZABILITY OF REPRESENTATIONS IN CYCLOTOMIC FIELDS

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ABSTRACT. In this paper sufficient conditions are given for a real valued complex irreducible character χ to be written in $Q(\chi)(\varepsilon_q)$, ε_q a primitive q th root of unity, q a prime dividing the order of the group.

Let T be an irreducible complex representation of the finite group G and suppose that the character χ of T is real valued. In [3, Theorem 3, p. 712], R. Brauer proved that T can be written in $Q(\chi)(\varepsilon_n)$, the field generated over $Q(\chi)$ (Q the rationals) by a primitive n th root of unity ε_n , where n divides the order $|G|$ of G and is either an odd prime or a power of 2. In this note we give a sufficient condition for T to be written in $Q(\chi)(\varepsilon_q)$ with q a prime divisor of $|G|$.

THEOREM. *Let T be an irreducible complex representation of the finite group G and suppose that the character χ of T is real valued. Assume there is a prime divisor q of $|G|$ and an element $\alpha \in G$ of order q such that $T(\alpha)$ does not have 1 as an eigenvalue, $2(q-1) \nmid \chi(1)$, and $Q(\chi) \cap Q(\varepsilon_q) = Q$. Then T can be written in $Q(\chi)(\varepsilon_q)$.*

PROOF. By the Brauer-Speiser theorem [3, p. 711] the Schur index of T over Q is 1 or 2. Since the result is clear if the Schur index is 1, we may assume that T has Schur index 2. In particular, $\chi(1)$ is even so q is odd. Let $K = Q(\chi)$ and let S be the K -representation of G such that $S \sim 2T$. Since $T(\alpha)$ does not have 1 as an eigenvalue, neither does $S(\alpha)$. Let $m = 2\chi(1)$ and set $A = S(\alpha)$. Denote by $K[A]$ the subring of the $m \times m$ matrices over K , $M_m(K)$, generated by A and the $m \times m$ scalar matrices. $K[A]$, being a homomorphic image of the group algebra over K of the cyclic group of order q , is a commutative semisimple ring. Hence $K[A]$

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is a field (isomorphic to $K(\varepsilon_q)$) if and only if it has no zero divisors. Let $f(x) = x^{q-1} + x^{q-2} + \cdots + x + 1$ and suppose P is an invertible $m \times m$ complex matrix with $P^{-1}AP$ diagonal. Since $A^q = I$ and 1 is not an eigenvalue, $f(P^{-1}AP) = P^{-1}f(A)P = 0$. Since $K \cap Q(\varepsilon_q) = Q$, $f(x)$ is irreducible in $K[x]$ and so $f(x)$ is the minimal polynomial of A . If $g(x)$ and $h(x)$ are in $K[x]$ with $g(A)h(A) = 0$, then $f(x)$ divides $g(x)h(x)$ and, by the irreducibility of $f(x)$, $f(x)$ divides $g(x)$ or $h(x)$. Thus $g(A)$ or $h(A)$ equals 0 and so $K[A] \cong K(\varepsilon_q)$. Now $K[A]$ is a subalgebra of the enveloping algebra of S . By the theory of the Schur index (see [4, §70]) this enveloping algebra is isomorphic to $M_n(D)$ where D is a division algebra central over K , $[D:K] = 4$ and $n = \chi(1)/2$. Thus $M_n(D) \supset K(\varepsilon_q)$. T can be written in $K(\varepsilon_q)$ if and only if $D \otimes_K K(\varepsilon_q) \sim K(\varepsilon_q)$. If this was not the case, then $D \otimes_K K(\varepsilon_q)$ would have index 2 and $M_{q-1}(D)$ would be the algebra of least degree similar to D and containing a subfield isomorphic to $K(\varepsilon_q)$ [1, Theorem 23, p. 60]. In this case we would also have $q-1 \mid n$ and so $2(q-1) \mid \chi(1)$, a contradiction. Thus $K(\varepsilon_q)$ splits D , proving the theorem.

REMARKS. (1) If $T(\alpha)$ does not have 1 as an eigenvalue then $(q-1) \mid \chi(1)$. For, since $[K(\varepsilon_q):K] = q-1$ and $\chi(\alpha) \in K$, every distinct eigenvalue of $T(\alpha)$ occurs with equal multiplicity and each ε_q^i , $1 \leq i \leq q-1$, occurs. We also note that if $\chi(1) = d(q-1)$ and $Q(\chi) \cap Q(\varepsilon_q) = Q$, then $T(\alpha)$ does not have 1 as an eigenvalue if and only if $\chi(\alpha) = -d$.

(2) Let G be generated by x , y , and c subject to the relations $x^7 = 1$, $y^{-1}xy = x^3$, $y^6 = c$, $c^2 = 1$, and c central. By results of [2] or [5] G has an irreducible complex representation T of Schur index 2 having a rational character χ of degree 6. Let S be the rational representation with $S \sim 2T$. The enveloping algebra of S is $M_3(D_{7,\infty})$ where $D_{7,\infty}$ denotes the rational division algebra with invariant $\frac{1}{2}$ at 7 and ∞ and invariant 0 at all other primes of Q . Let $q = 3$ and $\alpha = y^4$. Then $2(q-1) \nmid \chi(1)$ and $Q(\chi) \cap Q(\varepsilon_q) = Q$. Since 7 splits in $Q(\varepsilon_3)$, $Q(\varepsilon_3)$ does not split $D_{7,\infty}$ [1, Chapter 9]. Thus T cannot be written in $Q(\varepsilon_3)$. This shows that the hypothesis that 1 is not an eigenvalue cannot be dropped even if we assume $q-1 \mid \chi(1)$. (In this case $\chi(y^4) = 0$ so 1 is clearly an eigenvalue.)

(3) S_4 has a faithful absolutely irreducible rational representation T of degree 3. The enveloping algebra of this representation is $M_3(Q)$. Since $[Q(\varepsilon_3):Q] \nmid 3$, $M_3(Q)$ does not have a subalgebra isomorphic to $Q(\varepsilon_3)$ even though $T((123))^3 = 1$. The problem is that $T(123)$ has 1 as an eigenvalue and so $Q[T((123))]$ is not a field. More generally, Charles Ballantine has observed that if $A \in M_n(K)$, then $K[A]$ is a field if and only if its minimal polynomial is irreducible in $K[x]$. If $A \in M_n(Q)$ and $A^q = I$, q a prime, then $(A-1)(A^{q-1} + A^{q-2} + \cdots + A + 1) = 0$. If $A \neq I$ and 1 is an eigenvalue of A , then neither factor equals 0 so $Q[A]$ contains zero divisors.

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