PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 1, March 1973

VECTOR BUNDLES OVER FINITE CW-COMPLEXES ARE ALGEBRAIC

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ABSTRACT. It is proved that for any finite CW-complex X there exists a ring A of continuous functions on X, and natural 1-1 correspondences between the finitely generated projective A-modules (resp. $A \otimes_{\boldsymbol{R}} \boldsymbol{C}$-modules), and the topological real vector bundles (resp. complex vector bundles) over X where A is Noetherian and has Krull-dimension equal to the topological dimension of X.

1. **Introduction.** In Bass [**1**] the strong analogy, between the Grothendieck group of topological vector bundles on a finite CW-complex and the Grothendieck group of finitely generated projective modules over a Noetherian ring of finite dimension, was emphasized. The aim of this paper is to give a natural explanation of this analogy, by exhibiting for each finite CW-complex X a Noetherian subring A of $C^{\circ}(X, \boldsymbol{R})$, the Krull-dimension of which equals the topological dimension of X, and such that the inclusion $A \subset C^{\circ}(X, \boldsymbol{R})$ induces isomorphisms (Corollary 2),

$$(1) \qquad K_0(A) \to KO(X), \qquad K_0(A \otimes_{\boldsymbol{R}} \boldsymbol{C}) \to K(X).$$

In fact, we prove more: There is a natural bijection between the set of isomorphism classes of projective A-modules of finite type and the set of isomorphism classes of topological real vector bundles of finite rank over X. The analogous statement with $A \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and complex vector bundles also holds (Corollary 1).

It was proved in [**6**] that, when X is a compact differentiable manifold, one may take A equal to $A(X)$, the ring of real algebraic functions on X, which is defined by an arbitrary choice of algebraic embeddings of the connected components of X into Euclidean spaces (in the sense of Nash [**7**]). The ring $A(X)$ is independent of the embeddings; but if one is only interested in the isomorphisms (1) and one is not concerned about uniqueness, then $A(X)$ may be replaced by some $\boldsymbol{R}$-algebra essentially of finite type.

Received by the editors May 1, 1972.

AMS (MOS) subject classifications (1970). Primary 13D15, 18F30, 55B15; Secondary 32C35.

Key words and phrases. Vector bundles, topological K-theory, algebraic K-theory, finite CW-complexes, real analytic functions.

In the general case of a finite CW-complex, however, one has to content oneself with real analytic functions, and neither uniqueness nor finiteness is obtained in this way.

2. **Notations and preliminaries.** Denote by $\mathcal{O}$ the sheaf of real analytic functions on $\boldsymbol{R}^n$, and let, for any closed subset M of $\boldsymbol{R}^n$, $\mathscr{A}(M)=\Gamma(M, \mathcal{O})$ be the ring of germs of real analytic functions on M.

A subset M of $\boldsymbol{R}^n$ is called *semianalytic*, when every point $x \in M$ has an open neighborhood U_x in $\boldsymbol{R}^n$ such that $M \cap U_x$ is a finite union of finite intersections of sets of the form

$$\{y \in U_x \mid f_j(y) > 0\} \quad \text{or} \quad \{y \in U_x \mid f_j(y) = 0\},$$

where the f_j are real analytic functions on U_x. The following theorem was proved by Frisch [**3**]:

PROPOSITION 1 (J. FRISCH). *If $M \subset \boldsymbol{R}^n$ is a compact semianalytic set, then the ring $\mathscr{A}(M)$ is Noetherian.*

We shall also need the following well-known results from algebraic topology:

LEMMA 1. *A finite, connected* CW-*complex of dimension n is homotopy equivalent to some finite (connected) polyhedron $P \subset \boldsymbol{R}^{2n+1}$.*

Any finite polyhedron $P \subset \boldsymbol{R}^m$ has a neighborhood N, such that P is a strong deformation retract of N.

The first statement in this lemma is proved e.g. in Spanier [**8**, p. 120]. The neighborhood N in the second statement may be taken as a "second derived neighborhood of P", as defined in Hudson [**4**, p. 50].

A finite polyhedron $P \subset \boldsymbol{R}^m$ is obviously a semianalytic compact set; thus the ring $\mathscr{A}(P)$ is Noetherian by Proposition 1, but it is too large for our purposes, so we replace it by the following homomorphic image:

PROPOSITION 2. *Let $P \subset \boldsymbol{R}^m$ be a finite connected polyhedron in general position and denote by $\mathscr{B}(P)$ the ring of (continuous) functions on P that are locally restrictions to P of real analytic functions defined on open subsets of $\boldsymbol{R}^m$. Then the natural restriction mapping* $\mathrm{res}: \mathscr{A}(P) \to \mathscr{B}(P)$ *is surjective.*

PROOF. The polyhedron P is a finite union of convex polyhedrons $P_1, \cdots, P_r$. Let E_i denote the affine linear subspace of $\boldsymbol{R}^m$ spanned by P_i, and set $E=E_1 \cup \cdots \cup E_r$. We say that P is in general position, if there are no inclusions between the E_i's. Then E is a closed coherent analytic subspace of $\boldsymbol{R}^m$, whence cohomologically trivial. The sheaf of real analytic functions on E is written $\mathcal{O}_E$.

Let $f: P \to \boldsymbol{R}$ be a function in $\mathscr{B}(P)$, and denote the restriction of f to P_i by f_i. The compact set P_i is a convex body in E_i, hence P_i equals the closure of its interior int P_i in E_i. Each point $x \in P_i$ has an open neighborhood U_x in $\boldsymbol{R}^m$ such that f_i equals $g_{i,x}$ on $U_x \cap E_i$, for some $g_{i,x} \in \Gamma(U_x, \mathcal{O})$, that is, f_i extends to a real analytic function on $U_x \cap E_i$. The union U_i of the $U_x \cap E_i$, $x \in P_i$, is an open neighborhood of P_i in E_i; therefore, since int $P_i \cap U_x \neq \varnothing$ for all $x \in P_i$, f_i extends to an analytic function g_i on U_i, by the property of unique analytic prolongation. We may assume that every U_i is the ε-neighborhood of P_i for a fixed $\varepsilon > 0$, thus $U_i \cap E_j = U_j \cap E_i$. So $U = U_1 \cup \cdots \cup U_r$ is an open subset of E, and $g_1, \cdots, g_r$ define an element $g \in \Gamma(U, \mathcal{O}_E)$. Since $\boldsymbol{R}^m$ is cohomologically trivial, g is the restriction to U of a real analytic function h defined over an open set $V \subseteq \boldsymbol{R}^m$ for which $U = V \cap E$. This completes the proof, since h defines an element in $\mathscr{A}(P)$ which, restricted to P, gives f.

The ring $\mathscr{A}(P)$ is a regular, Noetherian ring of Krull-dimension m ($P \subset \boldsymbol{R}^m$); see Langmann [5] for a general treatment of this type of ring. Since P is Stein (i.e. P has a fundamental system of neighborhoods in $\boldsymbol{C}^m$ consisting of open Stein submanifolds of $\boldsymbol{C}^m$), P is homeomorphic to the maximal spectrum Max$(\mathscr{A}(P))$ of $\mathscr{A}(P)$, endowed with the Zariski topology. The kernel of the restriction map res: $\mathscr{A}(P) \to \mathscr{B}(P)$ is the Jacobson radical of $\mathscr{A}(P)$, so P is also homeomorphic to Max$(\mathscr{B}(P))$.

PROPOSITION 3. *In addition to the above notations, denote the topological dimension of P by n. The Krull-dimension of $\mathscr{B}(P)$ equals n.*

PROOF. Set $B = \mathscr{B}(P) \otimes_{\boldsymbol{R}} \boldsymbol{C}$, and denote the complexification of E_i by F_i, i.e. F_i is the complex affine subspace of $\boldsymbol{C}^m$ spanned by P_i.

B is a finite étale extension of $\mathscr{B}(P)$, hence dim B = dim $\mathscr{B}(P)$. Since P is Stein, every element of B may be extended (uniquely) to an element of $\Gamma(U \cap F, \mathcal{O}_F)$, where U is some open neighborhood of P in $\boldsymbol{C}^m$, and $\mathcal{O}_F$ is the sheaf of holomorphic functions on F. Therefore $B = \text{inj lim}_U\, \Gamma(U \cap F, \mathcal{O}_F)$, where U ranges over all domains of holomorphy in $\boldsymbol{C}^m$ that contain B; and $\Gamma(U \cap F, \mathcal{O}_F)$ becomes a subring of B.

To every chain of prime ideals of finite length in B, one may find a U such that the intersection with $\Gamma(U \cap F, \mathcal{O}_F)$ defines a chain of finitely generated prime ideals in $\Gamma(U \cap F, \mathcal{O}_F)$ of the same length. Since $U \cap F$ is defined by a finitely generated ideal in $\Gamma(U, \mathcal{O}_U)$, $U \cap F$ is a complex Stein space, and the proposition now follows from:

LEMMA 2. *Let X be a complex Stein space of dimension n. Then the maximal length of a chain of finitely generated prime ideals in $\Gamma(X, \mathcal{O}_X)$ is n.*

PROOF. See [2].

3. **Results.**

THEOREM. *Let $P \subset \boldsymbol{R}^m$ be a finite connected polyhedron in general position.*

Then there is a natural bijection between the isomorphism classes of projective $\mathscr{B}(P)$-modules (resp. $\mathscr{B}(P) \otimes_{\boldsymbol{R}} \boldsymbol{C}$-modules) of finite type and the isomorphism classes of topological real (resp. complex) vector bundles over P of finite rank.

Before giving the proof, we mention two corollaries.

COROLLARY 1. *The analogue of the theorem holds with P replaced by a finite* CW-*complex X, and $\mathscr{B}(P)$ replaced by some Noetherian subring A of $C^\circ(X, \boldsymbol{R})$, which satisfies* Krull-dim(A)=top dim(X).

PROOF. A finite CW-complex X is locally connected, so the category of topological vector bundles over X is the sum of the corresponding categories for each connected component X_i of X; and if a ring A is a finite product of Noetherian rings A_i, then A is Noetherian, Krull-dim(A)= max$_i${Krull-dim(A_i)}, and the category of (finitely generated) projective A-modules is the sum of the corresponding categories for each A_i. We may therefore assume that X is connected, hence homotopy-equivalent to a finite polyhedron $P \subset \boldsymbol{R}^m$—that clearly may be chosen in general position—by Lemma 1, and the corollary follows from Theorem 2.

COROLLARY 2. *One has isomorphisms*

$$K_0(A) \xrightarrow{\sim} KO(X), \qquad K_0(A \otimes_{\boldsymbol{R}} \boldsymbol{C}) \xrightarrow{\sim} K(X)$$

with X and A as in Corollary 1.

PROOF. An immediate consequence of Corollary 1.

PROOF OF THE THEOREM. The proof of the complex case is a trivial modification of the proof of the real case, so we confine ourselves to the latter. Denote by $\tilde{\mathscr{B}}$ the sheaf induced on $P = \operatorname{Max}(\mathscr{B}(P))$ from the affine scheme $\operatorname{Spec}(\mathscr{B}(P))$. It was proved in [**6**, Lemma 3.2] that one has a bijection:

$$H^1(\operatorname{Spec}(\mathscr{B}(P)), \mathbf{GL}_n(\operatorname{Spec}(\mathscr{B}(P)))) \xrightarrow{\sim} H^1(P, \mathbf{GL}_n(\tilde{\mathscr{B}}))$$

for all $n \in \boldsymbol{N}$. Furthermore, if $\mathscr{C}$ denotes the sheaf of real-valued continuous functions on P and $C = C^\circ(P, \boldsymbol{R}) = \Gamma(P, \mathscr{C})$, one has a bijection

$$H^1(\operatorname{Spec}(C), \mathbf{GL}_n(\operatorname{Spec}(C))) \xrightarrow{\sim} H^1(P, \mathbf{GL}_n(\mathscr{C})).$$

Thus it suffices to prove that the inclusion $\mathscr{B}(P) \subset C$ induces a bijection

$$\varphi : H^1(P, \mathbf{GL}_n(\mathscr{B})) \xrightarrow{\sim} H^1(P, \mathbf{GL}_n(\mathscr{C})).$$

The set $H^1(P, \mathbf{GL}_n(\mathscr{C}))$ may be identified with the set of homotopy classes of continuous mappings from P into some real Grassmannian $\mathbf{G}$, and since every continuous mapping is homotopic to the restriction of a real analytic mapping defined in a neighborhood of P in $\boldsymbol{R}^m$ (see e.g. [**6**, Lemma 3.1] with "Nash" replaced by "real analytic"), the mapping φ is surjective.

In order to prove the injectiveness of φ, we must show that if two $\tilde{\mathscr{B}}$-vector bundles are topologically isomorphic then they are also $\tilde{\mathscr{B}}$-isomorphic. The proof is reduced by standard arguments (see e.g. [**6**]) to proving that any continuous section in a $\tilde{\mathscr{B}}$-bundle over P may be arbitrarily well approximated by a $\tilde{\mathscr{B}}$-section. The latter is an easy application of Weierstrass' approximation theorem, as in [**6**, Proposition 4.5]. This shows that φ is injective, and hence ends the proof of the theorem.

REMARK. The ring A obtained in Corollary 1 is not unique, as is seen by the following example: Let X consist of two line segments having one end in common. If one embeds X in $\boldsymbol{R}^2$ as a straight line segment, A becomes an integral domain, but if one embeds X as an "angle", A is not integral. It obviously does not help to shrink A to the ring of algebraic functions, as one did in the case of a manifold.

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