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IMBEDDING CLASSES AND n-MINIMAL COMPLEXES

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ABSTRACT. Algebraic and geometrical techniques are used to study examples (new and previously conjectured) of n-dimensional simplicial complexes which cannot be topologically imbedded in Euclidean $2n$-space, but each proper subcomplex of any of them can be imbedded in Euclidean $2n$-space.

1. **Introduction.** An n-minimal complex is an n-dimensional simplicial complex which is not imbeddable in R^{2n} but each of its proper subcomplexes is imbeddable in R^{2n}. In this note we study n-minimal complexes by combining the geometric approach of Grünbaum **[2]** and Zaks **[7]** with the algebraic approach of Wu **[5]**. The new results presented here include a suspension theorem for symmetric deleted products (Theorem 3.1), an affirmative answer to a conjecture of Zaks on the minimality of certain 2-complexes, and a new way of constructing minimal 2-complexes.

2. **Definitions.** By an n-complex we mean a topological space which carries the structure of a fixed n-dimensional simplicial triangulation. The deleted product of an n-complex K is defined to be

$$D_2(K) = \{(x_1, x_2) \in K \times K \mid x_1 \neq x_2\}.$$

The polyhedral deleted product of an n-complex K is defined to be

$$D_2'(K) = \{(x_1, x_2) \in K \times K \mid C_r(x_1) \cap C_r(x_2) = \varnothing\},$$

where $C_r(x)$ is the smallest closed simplex of K containing x. Let τ denote the self-homeomorphism of $D_2(K)$ or $D_2'(K)$ defined by $\tau(x_1, x_2) = (x_2, x_1)$; the antipodal map on the n-sphere S^n, $0 \leq n \leq \infty$, is also denoted by τ. The quotient spaces of $D_2(K)$, $D_2'(K)$, and S^n under the action of τ are denoted by $\Sigma_2(K)$, $\Sigma_2'(K)$, and P^n ($\Sigma_2(K)$ is called the symmetric deleted product of K). A function f between spaces of the form $D_2(K)$, $D_2'(K)$, or S^n is equivariant if $f \circ \tau = \tau \circ f$. For a finite n-complex K, $D_2'(K)$ is an

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equivariant deformation retract of $D_2(K)$ (cf. [5]), so $\Sigma_2'(K)$ is a deformation retract of $\Sigma_2(K)$. For any n-complex K there is a unique (up to equivariant homotopy) equivariant map $\bar{c}_K: D_2(K)\to S^\infty$ (cf. [3, Chapter 4]), the kth (mod 2)-imbedding class of K is defined by $\Phi_2^k(K)=c_K^*(w_k)\in H^k(\Sigma_2(K); Z_2)$ where w_k is the nonzero element of $H^k(P^\infty; Z_2)$ and $c_K:\Sigma_2(K)\to P^\infty$ is the map induced by $\bar{c}_x$. If $f: K\to K'$ is an imbedding, denote by $D_2(f): D_2(K)\to D_2(K')$ the map given by $D_2(f)(x_1, x_2)=(f(x_1), f(x_2))$; $D_2(f)$ is equivariant and induces $\Sigma_2(f):\Sigma_2(K)\to\Sigma_2(K')$. By the uniqueness of $\bar{c}_{K'}$, $\Sigma_2(f)^*(\Phi_2^k(K'))=\Phi_2^k(K)$. Since $D_2(R^n)$ is equivariantly homotopy equivalent to S^{n-1}, $\Phi_2^k(R^m)\neq 0$ iff $0\leqq k\leqq m-1$; so $\Phi_2^m(K)\neq 0$ implies K cannot be imbedded in R^m. Note also that $\Phi_2^k(S^m)\neq 0$ iff $0\leqq k\leqq m$. The cone CK over an n-complex K is obtained from $K\times[0, 1]$ by identifying $K\times\{1\}$ to a point. The suspension SK of an n-complex K is obtained from $K\times[-1, 1]$ by identifying $K\times\{-1\}$ and $K\times\{+1\}$ to separate points. The join $K * K'$ of two complexes K and K' is the quotient space of $K\times K'\times[0, 1]$ under the identifications of the form $(x_1, x_2, 0)\sim(x_1, x_2', 0)$ or $(x_1, x_2, 1)\sim(x_1', x_2, 1)$. We endow CK, SK, and $K * K'$ with the usual simplicial triangulations. We always use singular cohomology; the group of singular j-chains on K is denoted by $\Delta_j(K)$, and $\Delta(K)$ denotes the singular chain complex of K. Given $f: K\to K'$, $f_\#$ denotes the map induced on chains. The ring of integers mod 2 is denoted by Z_2.

3. **The suspension theorem.** *If K is a finite n-complex and $i\geqq 0$, there is an isomorphism $\sigma_K: H^i(\Sigma_2(K); Z_2)\to H^{i+1}(\Sigma_2(SK); Z_2)$. If $f: K\to K'$ is an imbedding and $Sf: SK\to SK'$ is the suspension of f, then $\sigma_K\circ\Sigma_2(f)^*=\Sigma_2(Sf)^*\circ\sigma_{K'}$.*

PROOF. Let G be the multiplicative group of order 2 with elements 1 and α, and let R be the integral group ring of G. We consider Z_2 a trivial R-module (i.e. $(m+n\alpha)x=(m+n)x$, $x\in Z_2$). $\Delta_j(D_2(K))$ has an R-module structure given by $(m+n\alpha)\cdot s=ms+n\tau_\#(s)$, $s\in\Delta_j(D_2(K))$. $\Delta_j(SD_2(K))$ has an R-module structure defined by $(m+n\alpha)s=ms+n\tau_\#(s)$ where $s\in\Delta_j(SD_2(K))$ and $\tau: SD_2(K)\to SD_2(K)$ is defined by $\tau([x_1, x_2, t])=[x_2, x_1, -t]$. Finally, $\Delta_j(CD_2(K))\oplus\Delta_j(CD_2(K))$ has an R-module structure given by $(m+n\alpha)\cdot(s_1, s_2)=m(s_1, s_2)+n(\tau_\#(s_2), \tau_\#(s_1))$ where $\tau: CD_2(K)\to CD_2(K)$ is defined by $\tau([x_1, x_2, t])=[x_2, x_1, t]$.

Define $\beta:\Delta_j(D_2(K))\to\Delta_j(CD_2(K))\oplus\Delta_j(CD_2(K))$ by $\beta(s)=(i_\#(s), i_\#(s))$ where $i: D_2(K)\to CD_2(K)$ is given by $i(x_1, x_2)=[x_1, x_2, 0]$. Define $\gamma:\Delta_j(CD_2(K))\oplus\Delta_j(CD_2(K))\to\Delta_j(SD_2(K))$ by $\gamma(s_1, s_2)=j_{1\#}(s_1)-j_{2\#}(s_2)$ where, for $k=1, 2$, $j_k: CD_2(K)\to SD_2(K)$ is given by $j_k([x_1, x_2, t])=[(x_1, x_2), (-1)^{k-1}t]$. Denoting the duals of β and γ by $\beta^\#$ and $\gamma_\#$, a straightforward verification and a standard excision argument show that we have a short exact sequence of integral chain complexes

$$(*)\qquad \begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(\Delta(SD_2(K)); Z_2)\\ &\xrightarrow{\gamma\#} \operatorname{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); Z_2)\\ &\xrightarrow{\beta\#} \operatorname{Hom}_R(\Delta(D_2(K)); Z_2) \longrightarrow 0. \end{aligned}$$

Hence there is a long exact sequence

$$\begin{aligned} \cdots &\longrightarrow H^k(\operatorname{Hom}_R(\Delta(SD_2(K)); Z_2))\\ &\xrightarrow{\gamma*} H^k(\operatorname{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); Z_2))\\ &\xrightarrow{\beta*} H^k(\operatorname{Hom}_R(\Delta(D_2(K)); Z_2))\\ &\xrightarrow{\sigma'} H^{k+1}(\operatorname{Hom}_R(\Delta(SD_2(K)); Z_2)) \longrightarrow \cdots. \end{aligned}$$

Define $g: SD_2'(K) \to D_2'(SK)$ by $g([x_1, x_2, t]) = ([x_1, t], [x_2, -t])$. Then g is an equivariant homotopy equivalence (cf. **[1]**) with equivariant homotopy inverse $\tilde{g}: D_2'(SK) \to SD_2'(K)$ given by

$$\begin{aligned} \psi([x_1, t_1], [x_2, t_2]) &= [x_1, x_2, t_1] \quad \text{if } t_1 \geqq \max\{0, -t_2\} \text{ or } t_1 \leqq \min\{0, -t_2\},\\ &= [x_1, x_2, -t_2] \quad \text{if } -t_2 \geqq t_1 \geqq 0 \text{ or } -t_2 \leqq t_1 \leqq 0. \end{aligned}$$

Since the inclusions $D_2'(K) \to D_2(K)$ and $D_2'(SK) \to D_2'(SK)$ are equivariant homotopy equivalences, we have, from Proposition IV, 11.4, of **[4]**, isomorphisms

$$\begin{aligned} \lambda_1: H^k(\operatorname{Hom}_R(\Delta(SD_2(K)); Z_2)) &\cong H^k(\operatorname{Hom}_R(\Delta(D_2(SK)); Z_2))\\ &\cong H^k(\Sigma_2(SK); Z_2),\\ \lambda_2: H^k(\operatorname{Hom}_R(\Delta(D_2(K)); Z_2)) &\cong H^k(\Sigma_2(K); Z_2), \end{aligned}$$

and

$$\lambda_3: H^k(\operatorname{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); Z_2)) \cong H^k(CD_2(K); Z_2).$$

Since $H^k(CD_2(K); Z_2) = 0$ for $k > 0$ and both $CD_2(K)$ and $\Sigma_2(SK)$ are connected,

$$\sigma_K = \lambda_1^{-1}\sigma' \circ \lambda_2: H^k(\Sigma_2(K); Z_2) \to H^{k+1}(\Sigma_2(SK); Z_2)$$

is the desired isomorphism. The naturality of σ_K follows from the naturality of the short exact sequence $(*)$, the naturality of the λ_j's, and routine verifications.

3.1. COROLLARY. *If K is a finite n-complex, then $\Phi_2^k(K) = 0$ if and only if $\Phi_2^{k+1}(SK) = 0$.*

PROOF. Let $f: K \to S^{2n+1}$ be an imbedding. Since $\Phi_2^k(S^{2n+1})$ and $\Phi_2^{k+1}(S^{2n+2})$ are the unique nonzero elements of $H^k(\Sigma_2(S^{2n+1}); Z_2)$ and

$H^{k+1}(\Sigma_2(S^{2n+2}); Z_2)$ we have $\sigma(\Phi_2^k(S^{2n+1}))=\Phi_2^{k+1}(S^{2n+2})$. So

$$\begin{aligned}\sigma(\Phi_2^k(K)) &= \sigma \cdot \Sigma_2(f)^*(\Phi_2^k(S^{2n+1})) = \Sigma_2(Sf)^* \circ \sigma(\Phi_2^k(S^{2n+1})) \\ &= \Sigma_2(Sf)^*(\Phi_2^k(S^{2n+2})) = \Phi_2^{k+1}(SK).\end{aligned}$$

The corollary follows, since σ is an isomorphism.

4. **The classical n-minimal complexes.** Let K_{2n+3}^n be the complete n-complex on $2n+3$ vertices, i.e. the n-complex with $2n+3$ vertices in which every set of $n+1$ vertices spans an n-simplex. Then any complex of the form

$$(**) \qquad K = K_{2n_1+3}^{n_1} * K_{2n_2+3}^{n_2} * \cdots * K_{2n_p+3}^{n_p}$$

is an n-minimal complex where $n=n_1+n_2+\cdots+n_p+p-1$ (cf. [**2**]). In this section we give a new proof that $\Phi_2^{2n}(K)\neq 0$ whenever K has form $(**)$. Indeed Grünbaum proved in [**2**] that if K has the form $(**)$ then "there is a homeomorphism between $\hat{K}$ and S^{2n+1} which preserves antipodes". Converting this to our notation, Grünbaum's $\hat{K}$ is exactly our $D_2'(CK)$ and his homeomorphism preserving antipodes give us an equivariant homeomorphism

$$\phi' : D_2'(CK) \to S^{2n+1}$$

and hence an equivariant homotopy equivalence

$$\phi : D_2(CK) \to D_2(S^{2n+1}).$$

So, on quotient spaces, we have a homotopy equivalence

$$\psi : \Sigma_2(CK) \to \Sigma_2(S^{2n+1}).$$

Therefore $\Phi_2^{2n+1}(CK)=\psi^*(\Phi_2^{2n+1}(S^{2n+1}))\neq 0$. Since $CK\subseteq SK$, we have $\Phi_2^{2n+1}(SK)\neq 0$, and hence, by Corollary 3.1, $\Phi_2^{2n}(K)\neq 0$ as desired.

5. **The n-minimal complexes of Zaks.** In [7], Zaks proved the existence, for each $n\geqq 2$, of infinitely many mutually nonhomeomorphic n-minimal complexes. He was able to give explicit examples for $n>2$, but for $n=2$ a slight indeterminacy remained. In this section we remove that indeterminacy (exactly as Zaks conjectured it would be removed). Our main tool is

5.1. THEOREM. *Suppose K and K' are complexes and $\Phi_2^j(K)\neq 0$. If there is a continuous function $f: K\to K'$ such that for each $x\in K'$, $f^{-1}(x)$ is contained in a closed simplex of K, then $\Phi_2^j(K')\neq 0$.*

PROOF. Define $\phi_f: D_2'(K)\to D_2(K')$ by $\phi_f(x_1, x_2)=(f(x_1), f(x_2))$. Let r be an equivariant retraction of $D_2(K)$ onto $D_2'(K)$, and $\lambda: \Sigma_2(K)\to\Sigma_2(K')$ be the map induced on quotient spaces by $\phi_f\circ r: D_2(K)\to D_2(K')$. Then $\lambda^*(\Phi_2^j(K'))=\Phi_2^j(K)\neq 0$. So $\Phi_2^j(K')\neq 0$.

5.2. *Modified Zaks construction.* Consider the sequence of 2-complexes $X_0, X_1, X_2, \cdots$, where $X_0 = K_7^2$ and X_j is constructed from X_{j-1} as follows: let x_j and y_i be distinct points in the interior of the same 2-simplex of X_{j-1}; subdivide X_{j-1} so that x_j and y_j are nonadjacent vertices of the new triangulation; then X_j is the quotient complex of X_{j-1} obtained by identifying x_j and y_j. Applying Theorem 5.1 to the natural projection map $p_j: X_{j-1} \to X_j$ we have $\Phi_2^4(X_j) \neq 0$, and so X_j is not imbeddable in R^4, for each $j \geqq 0$. Zak's argument now completes the proof that X_j is in fact 2-minimal. Since X_j has exactly j local cut-points, X_i and X_j are not homeomorphic if $i \neq j$.

6. **More 2-minimal complexes.** In this section we describe a simple procedure for constructing many new 2-minimal complexes. The procedure can be adapted to one for constructing n-minimal complexes for $n > 2$. Our examples show that the collection of 2-minimal complexes is not nearly exhausted by repeatedly applying Zaks construction to one of the complexes K_7^2, $K_5^1 * K_3^0$, or $K_3^0 * K_3^0 * K_3^0$. Let T be a tree (finite contractible 1-complex) and f_1, f_2 be simplicial imbeddings of T into a subdivision of $K = K_7^2$ such that $f_1(T) \cap f_2(T) = \varnothing$ and $f_1(T) \cup f_2(T)$ is a subset of the interior of a 2-simplex of the original triangulation of K. Let L be the quotient complex obtained by identifying $f_1(t)$ with $f_2(t)$ for each $t \in T$. By Theorem 5.1, $\Phi_2^4(L) \neq 0$, so L is not imbeddable in R^4. To show that L is 2-minimal, let Δ be a 2-simplex of L. Then Δ is a 2-simplex of K and it suffices to consider the case $\Delta \cap (f_1(T) \cup f_2(T)) = \varnothing$. Set $K' = K - \operatorname{int} \Delta$ and $L' = L - \operatorname{int} \Delta$, and let $i: K' \to R^4$ be a piecewise linear imbedding (cf. **[6]**). We take R^4 to be the space of quadruples (x_1, x_2, x_3, x_4). By a deformation of K' we can assume there is a 2-simplex S of the subdivided K' and 2-disks D_1 and D_2 in the interior of S such that $f_i(T) \subseteq D_i$, $i = 1, 2$, and i is linear on S. We now alter i so that $i(S)$ is contained in the $x_4 = 0$ hyperplane of R^4. Now alter i again so that

$$i(D_1) = \{(x_1, x_2, 0, 0) \in R^4 \mid x_1^2 + x_2^2 = 1\},$$
$$i(D_2) = \{(x_1, x_2, 0, 1) \in R^4 \mid x_1^2 + x_2^2 = 1\},$$

and

$$\{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 = 1, 0 \leqq x_4 \leqq 1\} = i(D_1) \cup i(D_2).$$

Now assume T is a subcomplex of the standard 3-ball D^3. Since any two piecewise linear imbeddings of a tree in R^3 are ambiently isotopic, there is an imbedding $h: D^3 \times [0, 1] \to R^4$ such that $\pi_4 h(x_1, x_2, x_3, s) = s$ where $\pi_4(x_1, x_2, x_3, x_4) = x_4$;

$$h(x_1, x_2, x_3, s) = (x_1, x_2, x_3, s)$$

if $x_1^2+x_2^2+x_3^2=1$, $0\leqq s\leqq 1$;

$$h(t, 0) = i \circ f_1(t) \quad \text{if } t \in T;$$

and

$$h(t, 1) = i \circ f_2(t) \quad \text{if } t \in T.$$

Let $g: D^3 \to [0, 1]$ be a piecewise linear map such that $g(x_1, x_2, x_3)=0$ iff $(x_1, x_2, x_3) \in T$ and $g(x_1, x_2, x_3)=1$ iff $x_1^2+x_2^2+x_3^2=1$. Define $k: D_2 \to D^3$ by $k(x)=\pi_1 \circ h^{-1} \circ i(x)$ where $\pi_1: D_3 \times [0, 1] \to D^3$ is the projection. Finally define $j: K' \to R^4$ by

$$\begin{aligned} j(x) &= i(x) && \text{if } x \in K' - \operatorname{int} D_2, \\ &= h(K(x), g(K(x))) && \text{if } x \in D_2. \end{aligned}$$

It is easily verified that j induces an imbedding of L' in R^4, and our proof that L is 2-minimal is complete. L is distinct from any result of the Zaks construction since L has no local cut-points, and L is distinct from the classical 2-minimal complexes since L is not simply connected. By choosing T to be very complicated and iterating the above process, 2-minimal complexes of great complexity can be constructed.

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