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NORMAL SUBGROUPS OF FUCHSIAN GROUPS WITH FIXED PARABOLIC CLASS NUMBER

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ABSTRACT. Let F be a finitely generated Fuchsian group of the first kind. We provided necessary and sufficient conditions such that, for any $t\geqq 1$, F has only a finite number of normal subgroups with t parabolic classes.

1. **Introduction.** In this paper we consider finitely generated Fuchsian groups F of the first kind. L. Greenberg [1] has shown that for any $t\geqq 1$ there are only a finite number of normal subgroups of the $(2, 3, \infty)$ modular group with t parabolic classes. M. Newman [3] has obtained the same result for any (p, q, ∞) triangle group where $(p, q)=1$. We shall now resolve this problem for an arbitrary Fuchsian group by providing necessary and sufficient conditions for this to occur.

2. **Preliminaries.** Let D denote the unit disc $\{z:|z|<1\}$. A *Fuchsian* group F is a discrete subgroup of $L(D)$, the group of conformal homeomorphisms of D. A Fuchsian group F is said to be of the *first kind* if the closure, of the set of fixed points of nonelliptic elements of F distinct from the identity, is the entire boundary of D.

Finitely generated Fuchsian groups of the first kind have the presentation:

Generators: $a_1 b_1, \cdots, a_g b_g, e_1, \cdots, e_k, p_1, \cdots, p_r$

Defining relations: $e_1^{\nu_1} = e_2^{\nu_2} = \cdots = e_k^{\nu_k} = 1$

$$\left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) e_1, \cdots, e_k, p_1, \cdots, p_r = 1.$$

If F has the above presentation, we say that F has *signature* $(g; \nu_1, \cdots, \nu_k; r)$ and denote the group by $F(g; \nu_1, \cdots, \nu_k; r)$. The signature tells us there are precisely k nonconjugate (in F) maximal cyclic subgroups of

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[1] These results are contained in the author's doctoral thesis which was written under the direction of Professor Leon Greenberg at the University of Maryland.

F generated by an elliptic element of order ν_i, $i=1, \cdots, k$, and r nonconjugate (in F) maximal cyclic subgroups of F generated by a parabolic element. F is said to have *r parabolic classes*. The letter g denotes the genus of F.

The following theorem (Knopp and Newman **[2]**) will be useful.

THEOREM 1. *Let N be a normal subgroup of a Fuchsian group F with finite index μ. Suppose that the r parabolic generators in F have exponent m_i modulo N, $1 \leqq i \leqq r$ (that is m_i is least positive integer such that $p_i^{m_i} \in N$). Then the number t of parabolic classes of N is given by $t=\mu \sum_{i=1}^{r} (1/m_i)$.*

PROOF. Since F acts discontinuously on D we obtain quotient surfaces D/N and D/F. N being a normal subgroup of F implies the (branched) covering $\phi: D/N \rightarrow D/F$ is normal. D/F is a closed surface with r punctures and D/N is a closed surface with μ/m_i punctures over each of the r punctures in D/F. The theorem now follows. □

3. Initial results.

LEMMA 2. *Let p and q be distinct primes with the properties $q>p$ and $q \equiv 1 \pmod p$. There exists a nonabelian group of order pq.*

PROOF. Let $H=\langle b \rangle$, $K=\langle a \rangle$ where $a^p=b^q=1$. Since $q \equiv 1 \pmod p$ we can, by the Fermat theorem, choose an integer r such that $r \not\equiv 1 \pmod q$ but $r^p \equiv 1 \pmod q$. Define $\psi(a^i)b=b^{r^i}$, $1 \leqq i \leqq p$. Then $\psi(a^i)b$ is clearly an automorphism of H and ψ is a homomorphism from K to the automorphism group of H. Therefore we can form the semidirect product G of H by K.

Since

$$|G| = |HK| = (|H| \cdot |K|)/|H \cap K| = pq$$

G has order pq. Since $\psi(a)b=a^{-1}ba=b^r$, $r \not\equiv 1 \pmod q$ we conclude that G is nonabelian. □

THEOREM 3. *There exists an infinite number of groups G of order pn, p prime, n a positive integer, with generators satisfying the relations*

$$x^p = y^p = (xy)^n = z^n = 1.$$

PROOF. Let G be the nonabelian group of order pq in Lemma 2. Since $q>p$ there is a unique q-Sylow subgroup generated by some element $z \in G$. Let a generate one of the p-Sylow subgroups. Consider the element $y=az$. Then y cannot have order q. For, if it did, the uniqueness of our q-Sylow subgroup would imply $az=z^k$ for some $1 \leqq k<p$ and hence $a=z^{k-1}$. But a has order p and z has order q, therefore $\langle a \rangle$ and $\langle z \rangle$ can only have the identity element in common. Hence y must have order p (y cannot have order pq since G is noncyclic). Setting $x=a^{-1}$ we obtain the

relations $x^p=y^p=(xy)^q=1$. Clearly x and y generate G. By the Dirichlet theorem on primes there are an infinite number of primes q satisfying $q\equiv 1 \pmod p$, and the theorem follows. $\square$

We shall denote the groups of the previous theorem by $[p, p, n]$.

Let G be a group with an abelian subgroup A of finite index n. Let $X=\{x_1, x_2, \cdots, x_n\}$ be a set of coset representatives of the right cosets Ag. If $g \in G$ we define a mapping τ of G into A by

$$\tau(g) = x_1g(\overline{x_1g})^{-1}x_2g(\overline{x_2g})^{-1} \cdots x_ng(\overline{x_ng})^{-1}$$

where $\bar{g} \in X$ is the representative of the right coset Ag. The mapping τ is a homomorphism of G into A called the *transfer* of G into A. If A is contained in the center of G, $\tau(g)=g^n$.

4. Main result.

DEFINITION. A Fuchsian group F is said to have the *finite class property* (FCP) if for any integer $t\geqq 1$ there are only a finite number of normal subgroups of F with t parabolic classes.

THEOREM 4. *A Fuchsian group F has* (*FCP*) *if and only if* (1) *F is of genus zero*, (2) *F has exactly one parabolic class and* (3) *the elliptic generators of F have pairwise relatively prime orders. That is if and only if $F=F(0; \nu_1, \nu_2, \cdots, \nu_k; 1)$, $(\nu_i, \nu_j)=1$, $i\neq j$, $i, j=1, \cdots, k$.*

PROOF. We shall prove the necessity first. Our method is to show that if F is not of the above type we can construct for some $t\geqq 1$ an infinite number of normal subgroups with t parabolic classes.

Case 1. We first show that F cannot have more than one parabolic class.

Suppose F has two or more parabolic classes $p_1, p_2, \cdots, p_r, p$ with relation:

$$(*) \qquad \left(\prod_{i=1}^{g} a_ib_ia_i^{-1}b_i^{-1}\right)e_1, \cdots, e_k, p_1, \cdots, p_r = p.$$

Denote by C_q the cyclic group of prime order q generated by x. Define the homomorphism $\phi: F\to C_q$ by

$$\phi(e_j) = \phi(a_i) = \phi(b_i) = 1, \qquad 1 \leqq j \leqq k,\ 1 \leqq i \leqq g,$$
$$\phi(p_1) = x^{m_1}, \cdots, \phi(p_r) = x^{m_r}, \phi(p) = x,$$

where $1\leqq m_i<q$, $\sum_{i=1}^{r} m_i\equiv 1 \pmod q$.

Since q is prime the order of x^{m_i} is q, $i=1, \cdots, r$. Let $N=\ker \phi$. Then p_i, p, $i=1, \cdots, r$, have exponent q modulo N. By Theorem 1, N has $r+1$ parabolic classes. Letting $q\to\infty$ we have obtained an infinite number of normal subgroups N with $t=r+1$ parabolic classes.

Case 2. Suppose F has positive genus g. Consider the homomorphism $\phi: F \to [2, 2, n]$ defined by

$$\phi(a_1) = x, \quad \phi(b_1) = z, \quad \phi(p) = z^{-2} \quad \text{and}$$
$$\phi(a_j) = \phi(b_j) = \phi(e_i) = 1, \qquad 1 \leqq i \leqq k, 2 \leqq j \leqq g.$$

This is in fact a homomorphism, for in $[2, 2, n] \simeq D_n$, the dihedral group on n letters, $xzx^{-1}=z^{-1}$ or $xzx^{-1}z^{-1}=z^{-2}$.

Let $n=2m$ and set $N=\ker \phi$. Then $\phi(p^m)=z^{-2m}=z^{-n}=1$ implies $p^m \in N$. Clearly $\mu=|F/N|=2 \cdot 2m=4m$. Therefore setting $t=4$ and letting $m \to \infty$ we have found an infinite number of normal subgroups with $t=4$ parabolic classes.

Case 3. If F has genus zero and only one parabolic class generator p, F must have at least two elliptic elements e_1, e_2. This is an immediate consequence of the relation (*). We shall now assume that there is a prime q dividing the orders ν_1 and ν_2 of e_1 and e_2 respectively. Define the homomorphism $\phi: F \to [q, q, n]$ by

$$\phi(e_1) = x, \quad \phi(e_2) = y, \quad \phi(p) = xy \quad \text{and} \quad \phi(e_j) = 1, \qquad 3 \leqq j \leqq k.$$

Setting $N=\ker \phi$ we have $p^n \in N$ and $|F/N|=qn$. Taking $t=q$ and letting $n \to \infty$ we obtain an infinite number of normal subgroups N with $t=q$ parabolic classes.

These are clearly the only cases and the necessity is completed.

To prove the sufficiency, the following lemma is used.

LEMMA 5. *Let F be a Fuchsian group with signature*

$$\sigma = (0; \nu_1, \nu_2, \cdots, \nu_k; 1)$$

where $(\nu_i, \nu_j)=1$, $i \neq j$, $i, j=1, 2, \cdots, k$. Let N be a normal subgroup of finite index in F with t parabolic generators. Let $G=F/N$. G has a subgroup U such that

(a) *$U \subset$ center of G,*

(b) *$[G:U] \leqq t^t$.*

PROOF. Let $\{x_i\}_{i=1}^k$, z be the images of the elliptic and parabolic generators in G. Let $U=\bigcap_{g \in G} g\langle z\rangle g^{-1}$. U is clearly a normal subgroup of G. For $g \in G$ let $\alpha_g: U \to U$ be the automorphism $(u)\alpha_g=g^{-1}ug$. We have the relations

$$(\dagger) \qquad \alpha_{x_1}^{\nu_1} = \alpha_{x_2}^{\nu_2} = \cdots = \alpha_{x_k}^{\nu_k} = 1,$$
$$\alpha_{x_1}\alpha_{x_2} \cdots \alpha_{x_k} = \alpha_z = 1.$$

Since U is cyclic its automorphism group is abelian and the α_{x_i}, $i=1, \cdots, k$, commute. Solving equation (†) for α_{x_j} in terms of the remaining α_{x_i}, $i\neq j$, $i=1, \cdots, k$, we conclude that the order of α_{x_j} divides $\nu_1\nu_2 \cdots \nu_{j-1}\nu_{j+1} \cdots \nu_k$. But the order of α_{x_j} divides ν_j and $(\nu_i, \nu_j)=1$, $i\neq j$, $i=1, \cdots, k$, and therefore α_{x_j} has order 1, that is $\alpha_{x_j}=1$. This procedure works for all α_{x_j}, $j=1, \cdots, k$, hence U is central in G. Let $|G|=\mu$ and $|\langle z\rangle|=n$. By Theorem 1,

$$[G:\langle z\rangle] = [G:g\langle z\rangle g^{-1}] = t,$$

and there are $[G:N\langle z\rangle]$ subgroups conjugate to $\langle z\rangle$ with

$$[G:N\langle z\rangle] \leqq [G:\langle z\rangle] \leqq t,$$

we conclude that $[G:U]\leqq t^t$.

We now return to the proof of our theorem. Let us consider the following homomorphism of F into U:

$$F \xrightarrow{\phi} F/N \xrightarrow{\tau} U.$$

The mapping ϕ is the canonical homomorphism and τ is the transfer homomorphism. Let $H=G/U$, $G=F/N$, $r=|U|$, $s=|H|$. Then $s=\mu/r$ and by the lemma $s\leqq t^t$. Let U^s denote the subgroup of U consisting of all sth powers.

Since $U\subset Z(G)$, for all $g\in G$, $\tau(g)=g^s$. Hence $\tau\circ\phi$ is an epimorphism of F onto $G^s\supset U^s$, $G^s\subset U$. Since an abelian homomorphic image of F has order at most $\nu_1\nu_2\cdots\nu_k$, we obtain $|U^s|\leqq|G^s|\leqq\nu_1\nu_2\cdots\nu_k$. As U is cyclic we get $r/s\leqq r/(r,s)=|U^s|$. Therefore $r\leqq s|U^s|$ or $r\leqq\nu_1\nu_2\cdots\nu_k t^t$.

Since $\mu=rs$ we conclude $\mu\leqq\nu_1\nu_2\cdots\nu_k t^{2t}$.

Since the number of subgroups of bounded index in a finitely generated group is finite (Hall [4]) we obtain our result. □

References

1. L. Greenberg, *Note on normal subgroups of the modular group*, Proc. Amer. Math. Soc. **17** (1966), 1195–1198. MR **33** #7423.

2. M. I. Knopp and M. Newman, *Congruence subgroups of positive genus of the modular group*, Illinois J. Math. **9** (1965), 577–583. MR **31** #5902.

3. M. Newman, *Two-generator groups and parabolic class numbers*, Proc. Amer. Math. Soc. **31** (1972), 51–53.

4. M. Hall, Jr., *The theory of groups*, Macmillan, New York, 1959, MR **21** #1996.

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