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SHORTER NOTES

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A SHORT PROOF OF THE MARTINGALE CONVERGENCE THEOREM

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ABSTRACT. The martingale convergence theorem is first proved for uniformly integrable martingales by a standard application of Doob's maximal inequality. A simple truncation argument is then given which reduces the proof of the L^1-bounded martingale theorem to the uniformly integrable case. A similar method is used to prove Burkholder's martingale transform convergence theorem.

1. **Introduction.** Doob's classical martingale convergence theorem states that if $\{X_n, \mathscr{F}_n, n \geqq 1\}$ is an L^1-bounded martingale on a probability space $(\Omega, \mathscr{F}, P)$, then $\lim_n X_n$ exists and is finite P-almost everywhere. Several different proofs of this result are now known. The purpose of this note is to present a particularly simple proof based on Doob's maximal inequality $P\{\sup_n |X_n| \geqq \lambda\} \leqq \sup_n E\{|X_n|\}/\lambda$, $\lambda > 0$. The reader is referred to [**2**] or [**5**] for background material.

2. **The convergence theorem.** The martingale $\{X_n, \mathscr{F}_n, n \geqq 1\}$ is called complete if there exists a random variable X with $E\{X | \mathscr{F}_n\} = X_n$ for $n \geqq 1$. There is no loss of generality in assuming that X is $\mathscr{F}_\infty$-measurable where $\mathscr{F}_\infty$ is the σ-field generated by $\bigcup \mathscr{F}_n$. It is an elementary exercise to show that a martingale is complete if and only if it is uniformly integrable. We remark only that the necessity is proved by defining a set function $\mu(A) = \lim_n E\{X_n; A\}$ on the field $\bigcup \mathscr{F}_n$, proving from the uniform integrability that μ is a finite signed measure on $\bigcup \mathscr{F}_n$ which is absolutely continuous with respect to P, and defining X as the Radon-Nikodym derivative $d\tau/dP$ where τ is the unique extension of μ to $\mathscr{F}_\infty$.

We first prove the convergence theorem for complete martingales. It suffices to show for every $\varepsilon > 0$ there is a convergent martingale $\{Y_n, \mathscr{F}_n, n \geqq 1\}$ such that $P\{\sup_n |X_n - Y_n| \geqq \varepsilon\} < \varepsilon$. The collection of all integrable

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random variables Y for which there exists an integer $n(Y)$ such that Y is $\mathscr{F}_{n(Y)}$-measurable is dense in $L^1(\Omega, \mathscr{F}, P)$. Choose such a Y with $E\{|X-Y|\}<\varepsilon^2$ and let $Y_n=E\{Y|\mathscr{F}_n\}$. The martingale $\{Y_n, \mathscr{F}_n, n\geqq 1\}$ is trivially convergent since $Y_n=Y$ if $n\geqq n(Y)$ and the maximal inequality gives $P\{\sup_n|X_n-Y_n|\geqq\varepsilon\}\leqq\sup_n E\{|X_n-Y_n|\}/\varepsilon\leqq E\{|X-Y|\}/\varepsilon<\varepsilon$.

The above proof is a well-known example of Banach's convergence principle (see [3]) and we have included it here only for the sake of completeness. The main point to be made here is that any L^1-bounded martingale $\{X_n, \mathscr{F}_n, n\geqq 1\}$ can be approximated by a uniformly integrable martingale in the sense described below. If $\lambda>0$ is given, let T be the first time that $|X_n|\geqq\lambda$. T is a stopping time and we claim that the martingale $\{X_{T\wedge n}, \mathscr{F}_n, n\geqq 1\}$ is uniformly integrable. To see this let $Z=|X_T|$ on the set $\{T<\infty\}$ and $=\lambda$ on the set $\{T=\infty\}$. Since $|X_{T\wedge n}|\leqq Z$, uniform integrability will follow once we show that $E\{Z\}<\infty$. Now $E\{Z;\, T=\infty\}\leqq\lambda$ and

$$\begin{aligned} E\{Z;\, T<\infty\} = E\{|X_T|;\, T<\infty\} &= E\left\{\lim_n |X_{T\wedge n}|;\, T<\infty\right\} \\ &\leqq \liminf_n E\{|X_{T\wedge n}|;\, T<\infty\} \\ &\leqq \sup_n E\{|X_{T\wedge n}|\} \leqq \sup_n E\{|X_n|\} < \infty. \end{aligned}$$

The proof is completed by observing that $\{X_n\}$ and $\{X_{T\wedge n}\}$ coincide for all $n\geqq 1$ except on the set where $\sup_n |X_n|\geqq\lambda$ and this set has small measure when λ is large by the maximal inequality.

3. **Martingale transforms.** In [1] Burkholder proved that if $\{W_n, \mathscr{F}_n, n\geqq 1\}$ is the martingale transform of an L^1-bounded martingale $\{X_n, \mathscr{F}_n, n\geqq 1\}$ by a uniformly bounded multiplier sequence, then $\lim_n W_n$ exists and is finite P-almost everywhere. Recently M. Rao [6] gave an elementary proof of the maximal inequality $P\{\sup_n|W_n|\geqq\lambda\}\leqq K\sup_n E\{|X_n|\}/\lambda$, where K is a constant depending only on the multiplier sequence. Luis Baez-Duarte [4] gave a proof of Burkholder's theorem based on the above maximal inequality. Burkholder's result can be proved from the maximal inequality by our methods by first proving that the transform of an L^2-bounded martingale is again an L^2-bounded martingale, approximating any complete martingale by an L^2-bounded martingale, and then using the above truncation to approximate any L^1-bounded martingale by a complete martingale.

Added in proof. A proof of the martingale convergence theorem using the Krickeberg decomposition theorem but otherwise similar to the above proof has recently appeared in *Martingales and stochastic integrals* by P. A. Meyer (Springer Lecture Notes, no. 284). This source also contains Doobs's elegant original proof which avoids the upcrossing inequality.

References

1. D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504. MR **34** #8456.

2. J. L. Doob, *Stochastic processes*, Wiley, New York; Chapman & Hall, London, 1953. MR **15**, 445.

3. A. M. Garsia, *Topics in almost everywhere convergence*, Lectures in Advanced Math., 4, Markham, Chicago, Ill., 1970. MR **41** #5869.

4. Luis Baez-Duarte, *On the convergence of martingale transforms*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **19** (1971), 319–322.

5. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, Mass., 1966. MR **34** #5119.

6. M. Rao, *Doob decomposition and Burkholder inequalities*, Séminaire de Probabilités VI, Springer-Verlag, Berlin and New York, 1972, pp. 198–201.

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