PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 1, March 1973

DUALITY IN B^*-ALGEBRAS

SHEILA A. McKILLIGAN

ABSTRACT. Let X be a locally compact Hausdorff space and let $C_0(X)$ be the algebra of continuous functions on X vanishing at infinity. Then $C_0(X)$ is a dual algebra if and only if the operator $\mu \to f\,d\mu$ is weakly completely continuous on $C_0(X)^*$ for all $f \in C_0(X)$. This improves a recent result of P. K. Wong and provides a description of dual B^*-algebras.

1. **Introduction.** In two recent papers ([**4**], [**5**]) the concept of duality for B^*-algebras has been considered. We offer a simple proof of an improved version of the theorem in [**5**] which characterises dual commutative B^*-algebras. In turn, this improves the corresponding result for noncommutative B^*-algebras.

The main idea involved is that of weakly completely continuous linear operator. A bounded linear operator T on a Banach space A is *weakly completely continuous* if, given $\{a_\alpha\}$ a bounded net in A, there exists $a \in A$ and a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ such that $\{Ta_\beta\}$ converges weakly to Ta. We abbreviate weakly completely continuous to w.c.c. If A is a Banach space we let $a \to \hat{a}$ be the natural isomorphism of A into A^{**}, the second dual of A, and write $A^\wedge = \{\hat{a} \in A^{**} : a \in A\}$.

If X is a locally compact Hausdorff space, we write $C(X)$, $C_0(X)$ and $M(X)$ for the bounded continuous functions on X, the continuous functions vanishing at infinity on X and the space of all bounded regular Borel measures on X, respectively. We have, of course, that $M(X) = C_0(X)^*$. For $f \in C_0(X)$, $\mu \in M(X)$, $f\,d\mu \in M(X)$ is defined by $f\,d\mu(g) = \int_X fg\,d\mu$ $(g \in C_0(X))$.

2. **Some results for B^*-algebras.** We prove some results for the algebra $C_0(X)$ where X is locally compact Hausdorff and note that they hold for any commutative B^*-algebra. The crucial point is that the weak complete continuity of certain operators on $M(X)$ is necessary and sufficient for X to be discrete.

THEOREM 1. *Let X be a compact Hausdorff space. If the operator $\mu \to f\,d\mu$ is w.c.c. on $M(X)$ for every $f \in C(X)$, then X is finite.*

Received by the editors June 8, 1972.
AMS (MOS) subject classifications (1970). Primary 46L05; Secondary 46J10.
Key words and phrases. Dual B^*-algebra, weakly completely continuous operators.

PROOF. We have, in particular, that the identity operator on $M(X)$ is w.c.c., which implies that the closed unit ball of $M(X)$ is weakly compact. Hence [2, p. 425], $M(X)$ is reflexive, and so [2, p. 67], $C(X)$ is reflexive. From this it follows that X is discrete and so finite.

We aim to adapt this result to locally compact X but to do this we require two technical lemmas which we now present.

LEMMA 2. *Suppose X is a locally compact Hausdorff space and Y a compact subset of X. There is an isometric linear space isomorphism of $M(Y)$ into $M(X)$ given by $\mu \to \mu'$ where*

$$\mu'(f) = \mu(f|_Y) \quad (f \in C_0(X)).$$

PROOF. This is immediate from the results on p. 133 of [3].

LEMMA 3. *Suppose X is a locally compact Hausdorff space and Y a compact subset of X. Suppose that $\mu \to f\,d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$. Then $\mu \to f\,d\mu$ is w.c.c. on $M(Y)$ for every $f \in C(Y)$.*

PROOF. Let $\{\mu_\alpha\}$ be a net in $M(Y)$ with $\|\mu_\alpha\| \leqq M$ for all α, and let $f \in C(Y)$. By Lemma 2, $\{\mu'_\alpha\}$ is a net in $M(X)$ with $\|\mu'_\alpha\| \leqq M$ for all α. Furthermore, by p. 133 of [3], there is $f' \in C_0(X)$ such that $f'|_Y = f$. From our weak continuity assumption, there is $\mu \in M(X)$ and a subnet $\{\mu_\beta\}$ of $\{\mu_\alpha\}$ with $\{f'\,d\mu'_\beta\}$ converging weakly to $f'\,d\mu$. If $\Phi: M(X) \to M(Y)$ is defined by restriction, then $\Phi^*: M(Y)^* \to M(X)^*$ is continuous and linear. Since $\Phi(\lambda') = \lambda$ and $\Phi(f'\,d\lambda') = f\,d\lambda$ ($\lambda \in M(Y)$), $\{f\,d\mu_\beta\}$ converges weakly to $f\,d\Phi(\mu)$, which establishes the result.

We are now able to prove our main theorem.

THEOREM 4. *Let X be a locally compact Hausdorff space such that $\mu \to f\,d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$. Then X is discrete.*

PROOF. If $Y \subset X$ is compact, then, by Lemma 4, $\mu \to f\,d\mu$ is w.c.c. on $M(Y)$ for every $f \in C(Y)$. Hence (Theorem 1), Y is finite. Since X is locally compact Hausdorff, X must be discrete.

It is this result which forms the backbone of our improved version of Wong's characterisation of dual commutative B^*-algebras, which is the equivalence of (i) and (iv) of our next theorem. We note that the condition $A' = A^*$ in (ii) of Theorem 3.2 of [5] is therefore unnecessary. With regard to (iii) we remark that the two Arens products coincide on $C_0(X)^{**}$ [1].

THEOREM 5. *For a locally compact Hausdorff space X the following are equivalent:*

(i) *$C_0(X)$ is a dual algebra;*
(ii) *X is discrete;*
(iii) *$C_0(X)^\wedge$ is an ideal of $C_0(X)^{**}$ with each Arens product;*
(iv) *$\mu \to f\,d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$.*

PROOF. The equivalence of (i) and (ii) is Theorem 4.2 of [4]. If (ii) holds, $M(X)=l_1(X)$ [3, p. 3] and we may identify $C_0(X)^{**}$ with $l_\infty(X)$, which is just $C(X)$. It is clear that the Arens product is the pointwise product in $C(X)$, and so $C_0(X)^\wedge$ is an ideal in $C_0(X)^{**}$.

We observe that for $f \in C_0(X)$, $\mu \in M(X)$, $f\,d\mu=\mu * f$ where $*$ is as in Wong's notation [5]. Since Lemma 2.2 of [5] depends only on the fact that if A is a dual B^*-algebra then $\pi(A)$ $(=A^\wedge)$ is an ideal of A^{**}, a very similar proof shows that (iii) implies (iv).

That (iv) implies (ii) is immediate from Theorem 4, and the proof is complete.

Thus Theorem 4.2 of [5] may be adjusted so that we have, using Wong's notation,

THEOREM 6. *If A is a B^*-algebra, then A is a dual algebra if and only if for every $x \in A$ the mapping $T_x: f \rightarrow f * x$ is weakly completely continuous on A^*.*

REFERENCES

1. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. **11** (1961), 847–870. MR **26** #622.

2. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

3. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #158.

4. B. J. Tomiuk and P. K. Wong, *The Arens product and duality in B^*-algebras*, Proc. Amer. Math. Soc. **25** (1970), 529–535. MR **41** #4256.

5. P. K. Wong, *The Arens product and duality in B^*-algebras.* II, Proc. Amer. Math. Soc. **27** (1971), 535–538. MR **43** #933.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720