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# THE NORM OF A DERIVATION IN A $W^*$-ALGEBRA

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ABSTRACT. The norm of an inner derivation $\delta_a$ of a (non-necessary separable) $W^*$-algebra $M$ is shown to satisfy

$$\|\delta_a\| = 2\inf\{\|a - z\|;\, z \in Z, \text{ the center of } M\},$$

and some related results are obtained.

Let $M$ be an associative algebra. A linear map $\delta: M \to M$ is called a derivation, if $\delta(xy) = x \cdot \delta(y) + \delta(x)y$ for all $x, y \in M$. A derivation $\delta$ is inner, if there exists $a \in M$, such that $\delta(x) = ax - xa$, $x \in M$. We denote by $\delta_a$ the inner derivation defined by $a$.

In [7] Sakai has shown that every derivation $\delta$ in a $W^*$-algebra $M$ is inner. Our aim is to find a "good" $a \in M$, such that $\delta = \delta_a$.

More precisely, we prove the following theorems:

THEOREM 1. *If $Z$ is the center of $M$, there exists a unique application $\Phi: M \to Z$, such that*

(i) $\Phi(za) = z\Phi(a)$, $z \in Z$, $a \in M$,

(ii) $\|a - \Phi(a)\| = \inf_{z \in Z}\|a - z\|$, $a \in M$.

*Furthermore*, $\Phi$ *is continuous in the norm topology.*

THEOREM 2. *With the notations from Theorem* 1,

$$\|\delta_a\| = 2 \cdot \|a - \Phi(a)\|.$$

If $\delta$ is a derivation on $M$, and $a \in M$ is such that $\delta = \delta_a$, then $a - \Phi(a)$ depends only on $\delta$. Put $a(\delta) = a - \Phi(a)$.

THEOREM 3. $\delta \mapsto a(\delta)$ *is a continuous mapping of the Banach space of all derivations on $M$ into $M$, equipped with the norm topology.*

These results are proved for the $W^*$-algebra $B(H)$ of all bounded linear operators in a Hilbert space $H$ by Stampfli [8]. We shall reduce the general problem to this one. Also another result from [8] may be extended for the case of an arbitrary $W^*$-algebra, using our reduction.

Theorems 1 and 2 imply the following

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COROLLARY. *If $a \in M$, then*

$$\|\delta_a\| = 2 \inf_{z \in Z} \|a - z\|.$$

This corollary is proved in [**6**] for selfadjoint $a$ and in [**3**] for $W^*$-algebras with a faithful representation in a separable Hilbert space.

1. **Preliminaries for the proofs.** Let $M$ be a $W^*$-algebra, $Z$ its center and $\Omega$ the maximal ideal space of $Z$. For every $t \in \Omega$, denote by $[t]$ the smallest norm-closed two-sided ideal of $M$ containing $t$. Let $M_t$ be the factor $C^*$-algebra $M/[t]$ and let $x_t$ denote the image of $x \in M$ in $M_t$. Glimm proved in [**4**] that for each $x \in M$ the function $t \mapsto \|x_t\|$ is continuous on $\Omega$.

Following a result of Halpern [**5**], $[t]$ is a primitive ideal for all $t \in \Omega$. Hence every $M_t$ has a faithful irreducible representation $\Pi_t$ in some Hilbert space $H_t$. If $a_t \in M_t$, the derivation $\delta_{\Pi_t(a_t)}$ on $\Pi_t M_t$ has a unique extension to a derivation in $B(H_t)$, and these two derivations have equal norms (see for example [**1**]).

By [**8**] we have the following lemma:

LEMMA. *For $a_t \in M_t$ and a complex number $\lambda_t$ the following statements are equivalent*:

(i) $\|a_t - \lambda_t\| = \inf_{\lambda \in C} \|a_t - \lambda\|$.

(ii) $\|a_t - \lambda_t\|^2 + |\lambda_t - \lambda|^2 \leqq \|a_t - \lambda\|^2$ *for all* $\lambda \in C$.

(iii) $\|\delta_{a_t}\| = 2 \cdot \|a_t - \lambda_t\|$.

*In particular, for every $a_t \in M_t$ there exists a unique $\lambda_t \in C$ such that the above equivalent conditions are satisfied. If $\|a_t' - a_t\| \leqq \varepsilon$ then $|\lambda_t' - \lambda_t| \leqq \frac{1}{2}(\varepsilon + (\varepsilon^2 + 8\varepsilon\|a_t - \lambda_t\|)^{1/2})$.*

2. **Proof of Theorems 1 and 2.** Let $a \in M$ and $a_t$ its canonical image in $M_t$. By the above Lemma, for every $t \in \Omega$ there exists a unique $\lambda_t \in C$ such that the statements of the Lemma hold.

Now, $t \mapsto \|a_t - \lambda_t\|$ is an upper semicontinuous function in $\Omega$. Indeed, if $\alpha > 0$ and $\|a_{t_0} - \lambda_{t_0}\| < \alpha$ for some fixed $t_0 \in \Omega$, then by Glimm's result there exists a neighborhood $V$ of $t_0$, such that, for $t \in V$, $\|a_t - \lambda_{t_0}\| < \alpha$. Hence for $t \in V$, $\|a_t - \lambda_t\| \leqq \|a_t - \lambda_{t_0}\| < \alpha$. So $\{t \mid t \in \Omega, \|a_t - \lambda_t\| < \alpha\}$ is open and the upper semicontinuity of $t \mapsto \|a_t - \lambda_t\|$ is proved.

Since $\Omega$ is hyperstonean, there exists an open dense set $D \subset \Omega$, such that the restriction of $t \mapsto \|a_t - \lambda_t\|$ to $D$ is continuous (see for example [**2**]).

Let $t_0 \in D$. By the Lemma, for every $t$,

$$\|a_t - \lambda_t\|^2 + |\lambda_t - \lambda_{t_0}|^2 \leqq \|a_t - \lambda_{t_0}\|^2.$$

Since $t_0$ is a continuity point of $t \mapsto \|a_t - \lambda_t\|$, $\lim_{t \to t_0} \|a_t - \lambda_t\| = \|a_{t_0} - \lambda_{t_0}\|$. On the other hand, by Glimm's result, $\lim_{t \to t_0} \|a_t - \lambda_{t_0}\| = \|a_{t_0} - \lambda_{t_0}\|$. Hence $\lim_{t \to t_0} |\lambda_t - \lambda_{t_0}| = 0$, that is $t \mapsto \lambda_t$ is continuous in $t_0$.

Using again the fact that $\Omega$ is hyperstonean, there exists a continuous function $f$ on $\Omega$ such that, on an open dense subset of $\Omega$, $f$ is given by $t \mapsto \lambda_t$.

If $t_0 \in \Omega$ is arbitrary, there exists a generalized sequence $(t_i)$, convergent to $t_0$, such that for every $i$, $f(t_i)=\lambda_{t_i}$. Obviously,

$$\|a_{t_i} - f(t_i)\| = \|a_{t_i} - \lambda_{t_i}\| \leqq \|a_{t_i} - \lambda_{t_0}\|.$$

But $f$ is a continuous function on $\Omega$, so it may be considered an element of $Z$, and by Glimm's result

$$\lim_i \|a_{t_i} - f(t_i)\| = \lim_i \|(a-f)_{t_i}\| = \|(a-f)_{t_0}\| = \|a_{t_0} - f(t_0)\|.$$

Again by Glimm's result

$$\lim_i \|a_{t_i} - \lambda_{t_0}\| = \|a_{t_0} - \lambda_{t_0}\|.$$

Hence

$$\|a_{t_0} - f(t_0)\| \leqq \|a_{t_0} - \lambda_{t_0}\|.$$

The converse inequality is obvious by the Lemma, and the unicity of $\lambda_{t_0}$ implies $f(t_0)=\lambda_{t_0}$.

In conclusion, $t \mapsto \lambda_t$ is everywhere equal to the continuous function $f$. Put $\Phi(a)=f$.

Since $\bigcap_{t\in\Omega} [t]=\{0\}$, for every $x \in M$, $\|x\|=\sup_{t\in\Omega} \|x_t\|$. Now it is easy to verify that $\Phi$ satisfies conditions (i) and (ii) of Theorem 1.

If $\Psi: M \to Z$ satisfies the conditions of Theorem 1, and there exists $a \in M$ such that $\Phi(a) \neq \Psi(a)$, then there exists a nonvoid open and closed set $V \subset \Omega$ and $\varepsilon > 0$, such that for $t \in V$, $|\lambda_t - \Psi(a)_t| = |\Phi(a)_t - \Psi(a)_t| \geqq \varepsilon$. If $z \in Z$ is the characteristic function of $V$, by condition (i) and the Lemma,

$$\begin{aligned}\|az - \Phi(az)\|^2 &= \sup_{t\in V} \|a_t - \lambda_t\|^2 \\ &\leqq \sup_{t\in V}(\|a_t - \Psi(a)_t\|^2 - |\lambda_t - \Psi(a)_t|^2) \\ &\leqq \|az - \Psi(az)\|^2 - \varepsilon^2\end{aligned}$$

in contradiction to condition (ii). Hence $\Psi=\Phi$.

The continuity of $\Phi$ results from the last statement of the Lemma.

Finally, if $x \in M$ and $\|x\| \leqq 1$, then for every $t \in \Omega$,

$$\|\delta_a(x)_t\| = \|\delta_{a_t}(x_t)\| \leqq 2\,\|a_t - \lambda_t\| \leqq 2\,\|a - \Phi(a)\|.$$

Hence

$$\|\delta_a(x)\| = \sup_t \|\delta_a(x)_t\| \leqq 2\,\|a - \Phi(a)\|,$$

and Theorem 2 is also proved.

3. **Proof of Theorem 3.** Using our construction of $\Phi$, it is easy to see that, for $a \in M$ and $z \in Z$, $\Phi(a+z)=\Phi(a)+z$. This implies that, in fact, $a-\Phi(a)$ depends only on $\delta_a$. Hence $\delta \mapsto a(\delta)$ is well defined.

Let $\delta'$ and $\delta$ be two derivations on $M$ such that $\|\delta'-\delta\| \leqq \varepsilon$. By the theorem of Sakai and Theorems 1 and 2 above, there exists $b \in M$, $\|b\| \leqq \varepsilon/2$ such that $\delta'-\delta=\delta_b$. If $a=a(\delta)$ and $a'=a+b$, then $\delta=\delta_a$, $\delta'=\delta_{a'}$ and $\|a'-a\| \leqq \varepsilon/2$. Using the construction of $\Phi$ and the Lemma,

$$\begin{aligned}\|\Phi(a') - \Phi(a)\| &\leqq \tfrac{1}{4}(\varepsilon + (\varepsilon^2 + 16\varepsilon \|a - \Phi(a)\|)^{1/2}) \\ &= \tfrac{1}{4}(\varepsilon + (\varepsilon^2 + 16\varepsilon \|a(\delta)\|)^{1/2}).\end{aligned}$$

Hence

$$\begin{aligned}\|a(\delta') - a(\delta)\| &\leqq \|a' - a\| + \|\Phi(a') - \Phi(a)\| \\ &\leqq \tfrac{3}{4}\varepsilon + \tfrac{1}{4}(\varepsilon^2 + 16\varepsilon \|a(\delta)\|)^{1/2}.\end{aligned}$$

This inequality implies the continuity of $\delta \mapsto a(\delta)$.

PROBLEM. What information about $M$ is given by $\Phi$?

We remark that $\Phi$ is not well understood even in the case $M=B(H)$ (see [**8**]).

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