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QUASI-UNMIXED LOCAL RINGS AND QUASI-SUBSPACES

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ABSTRACT. The concept of a quasi-subspace is defined so that it plays a role relative to quasi-unmixedness analogous to that of subspace to unmixedness. This definition is used to characterize quasi-unmixed local rings.

1. **Introduction.** In this paper, a ring shall be a commutative ring with identity. The terminology is basically that of [**3**] and [**9**]. In particular, a semilocal (Noetherian) ring R is called *unmixed* (resp., *quasi-unmixed*) in case depth p=altitude R, for every prime divisor (resp., minimal prime divisor) p of zero in the completion of R.

Proposition 3.3 in [**1**] gives an example of a local domain A of altitude two whose integral closure is a convergent power series ring in two variables over the complex number field, and whose completion A^* contains an imbedded prime divisor of zero. Thus A is not unmixed. However, by [**5**, Corollary 3.4(i)], A is quasi-unmixed.[2] (This example answers Problem 1 of [**2**, p. 62].)

Ratliff [**6**, §4] characterizes an unmixed local ring R in terms of certain local rings that contain R as a subspace. This paper parallels [**6**, §4]; in particular, the concept of a quasi-subspace is introduced to play a role relative to quasi-unmixedness analogous to the role played by a subspace to unmixedness. Since the concepts of unmixedness and quasi-unmixedness are distinct, the results and techniques below should be of assistance in investigating quasi-unmixed local rings. The results of this paper and of [**5**], [**6**] have been used in [**8**] to characterize unmixed and quasi-unmixed local domains. (Specifically, if $\mathscr{R}$ is a particular Rees ring of a local domain R, then the property that a certain transform ring of $\mathscr{R}$ is contained in the integral closure of $\mathscr{R}$ (resp., is Noetherian) is a condition which characterizes (resp., is closely related to) the quasi-unmixedness (resp., unmixedness) of R.)

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[1] This paper is part of the author's doctoral dissertation written at the University of California at Riverside under the direction of Professor Louis J. Ratliff, Jr.

[2] This example was mentioned to the author by Professor Louis J. Ratliff, Jr.

2. **Preliminary definitions and results.**

DEFINITION 1. Let R and S be semilocal rings with completions R^* and S^*. R is a *quasi-subspace* of S if there exists an isolated ideal component I^* of zero in R^* such that $I^* \subseteq \operatorname{rad} R^*$ and such that S^* dominates R^*/I^* and S dominates R/I, where $I = I^* \cap R$.

Note that $I^* \subseteq \operatorname{rad} R^*$ implies that $I \subseteq (\operatorname{rad} R^*) \cap R = \operatorname{rad} R$, and so $\operatorname{rad} I = \operatorname{rad} R$. Also, by letting $I^* = (0)$, note that a semilocal ring is a quasi-subspace of itself and that a semilocal ring that is a subspace of a semilocal ring S is also a quasi-subspace of S.

Lemma 2 below gives a characterization of quasi-subspace that is easier to use for the rings considered in §3. Lemma 3 then shows how the concept of quasi-subspace is related to the minimal prime divisors of zero of these rings. In particular, Lemma 3 and Corollary 8 give a relation between quasi-unmixed local rings and the minimal prime divisors of zero in certain Rees rings of their completions (Corollary 9).

For ease of notation, *let R_k denote a polynomial ring in k indeterminants over a ring R. For the completion R^* of R, $R_k^* = (R^*)_k$.*

LEMMA 2. *Let (R, M) be a local ring with completion (R^*, M^*). Let $k \geqq 0$, and let $y_1, \cdots, y_d$ $(d \geqq 0)$ be elements of the total quotient ring of R_k. Let $A = R_k[y_1, \cdots, y_d]$ and $A^* = R_k^*[y_1, \cdots, y_d]$. Let P^* be a prime ideal in A^* such that $P^* \cap R^* = M^*$, and let $P = P^* \cap A$. Then, R is a quasi-subspace of A_P if and only if $A^*_{P^*}$ dominates R^*/I^* for some isolated ideal component of zero in R^* such that $I^* \subseteq \operatorname{rad} R^*$.*

PROOF. Let $A^*_{P^*}$ dominate R^*/I^* and $I = I^* \cap R$, where I^* is given above. Let K (resp., K^*) be the kernel of the natural homomorphism of A into A_P (resp., A^* into $A^*_{P^*}$). Since $A_P = (A/K)_{P/K}$ is a dense subspace of $A^*_{P^*} = (A^*/K^*)_{P^*/K^*}$ [**6**, Lemma 3.2], then $K = K^* \cap A$. Also, $I^* = K^* \cap R^*$. Therefore, $I = K \cap R$, and so R/I is a subring of A_P. Since $P \cap R = M$ [**6**, Lemma 3.2], A_P dominates R/I. Since $(A_P)^* = (A^*_{P^*})^*$ [**6**, Lemma 3.2], and $(A^*_{P^*})^*$ dominates $A^*_{P^*}$, then $(A_P)^*$ must dominate R^*/I^*.

Conversely, let R be a quasi-subspace of A_P. Let I^* be as in Definition 1, and let K^* be as above. Then R^*/I^* is a subring of $(A_P)^* = (A^*_{P^*})^*$, and is therefore a subring of $A^*_{P^*}$. Hence $A^*_{P^*}$ dominates R^*/I^*, since $P^* \cap R^* = M^*$. Q.E.D.

LEMMA 3 (CF. [**6**, LEMMA 4.5(1)]). *Let R, R^*, A, A^*, P and P^* be as in Lemma 2. Then R is a quasi-subspace of A_P if and only if P^* contains all minimal prime divisors of zero in A^*.*

PROOF. Let R be a quasi-subspace of A_P. By Lemma 2, R^*/I^* is a subring of $A^*_{P^*}$, where I^* is given in Definition 1. Thus $I^* = K^* \cap R^*$, where K^* is given in Lemma 2. Therefore, since K^* is an isolated ideal component of zero in A^*, and since A^* and R_k^* have the same total quotient ring, it follows that $K^* \cap R_k^* = I^* R_k^* \subseteq (\text{rad } R^*) R_k^* = \text{rad } R_k^*$. Thus $(\text{rad } K^*) \cap R_k^* = \text{rad}(K^* \cap R_k^*) = \text{rad } R_k^*$, and so $\text{rad } K^* = \text{rad } A^*$. Hence P^* contains every minimal prime ideal in A^*.

Conversely, let K^* be as above, and define $I^* = K^* \cap R^*$. Then R^*/I^* is a subring of $A^*_{P^*}$. Since P^* contains all minimal prime ideals in A^*, $\text{rad } I^* = \text{rad}(K^* \cap R^*) = (\text{rad } K^*) \cap R^* = (\text{rad } A^*) \cap R^*$. Since R^* is a subring of A^*, $(\text{rad } A^*) \cap R^* = \text{rad } R^*$.

Also, since K^* is an isolated ideal component of zero in A^* and since R_k^* and A^* have the same total quotient ring, it follows that I^* is an isolated ideal component of zero in R^*. And $A^*_{P^*}$ dominates R^*/I^*, since $P^* \cap R^* = M^*$. Hence, by Lemma 2, R is a quasi-subspace of A_P. Q.E.D.

REMARK 4. We give a number of known properties of unmixed and quasi-unmixed semilocal rings that will be needed in the remainder of the paper:

(1) R is a quasi-unmixed semilocal ring if and only if R/q is quasi-unmixed and depth q = altitude R, for every minimal prime divisor q of zero in R [**4**, Lemma 2.2].

(2) If R is a quasi-unmixed semilocal ring and P is a prime ideal in R, then R_P is quasi-unmixed [**4**, Lemma 2.5].

(3) Let R be a semilocal domain. If R is quasi-unmixed and A is a finitely generated domain over R, then A is locally quasi-unmixed [**4**, Corollary 2.5].

(4) Let (R, M) be a local ring. If altitude $R=0$, or altitude $R=1$ and M is not a prime divisor of zero, then R is unmixed and, therefore, quasi-unmixed.

3. **Some characterizations of quasi-unmixed local rings.** With Lemma 3 and Remark 4, the techniques of [**6**] can be adapted to prove most of the following results. The proofs are essentially accomplished by replacing "subspace" by "quasi-subspace", "unmixed" by "quasi-unmixed", "prime divisor of zero" by "minimal prime divisor of zero" and "Remark 4.6" by "Remark 4", and by making the appropriate reference changes. Since the proofs of Corollary 7 and Corollary 8 are entirely analogous to those in [**6**], they will be omitted.

LEMMA 5 (CF. [**6**, LEMMA 4.5(2)]). *Let R, R^*, A and A^* be as in Lemma 2. Let P be a prime ideal of A such that R is a quasi-subspace of A_P. Then the following statements hold*:

(1) *$P^* = PA^*$ is a prime ideal of A^* that lies over P, and A_P is a dense subspace of $A^*_{P^*}$.*

(2) *R is quasi-unmixed if and only if A_P is quasi-unmixed.*

(3) *If Q is a prime ideal of A such that $P\subseteq Q$, then R is a quasi-subspace of A_Q.*

PROOF. By the domination of Definition 1, it is straightforward to show that $P\cap R=M$. (1) then follows by [**6**, Lemma 3.2]. It will be shown in Theorem 6(2)(a) that if R is quasi-unmixed, then A_P is quasi-unmixed (even if R is not a quasi-subspace of A_P). The converse of (2) can be shown by using the quasi-unmixedness of A^*_{P*}, (1) and Lemma 3 in an adaptation of the proof in [**6**]. (3) is easily proved by using Lemma 3. Q.E.D.

The following theorem is the main result of this paper. It will be applied (Corollary 8) to characterize a quasi-unmixed ring R in terms of quotient rings of certain Rees rings of R. Another application to a specific class of rings is given in Corollary 7.

THEOREM 6 (CF. [**6**, THEOREM 4.1]). *Let (R, M) be a local ring with* altitude $n\geqq 0$. *Then*:

(1) *R is quasi-unmixed if and only if there exist an integer k, elements $y_1, \cdots, y_d$ of the total quotient ring of R_k, and a prime ideal P in $A=R_k[y_1, \cdots, y_d]$ such that R is a quasi-subspace of A_P and A_P is quasi-unmixed.*

(2) *Let $f_0, f_1, \cdots, f_d$ be in R_k ($d\geqq 0$ and $k\geqq 0$), where f_0 is not a zero divisor in R_k. Let $y_i=f_i/f_0$ and $A=R_k[y_1, \cdots, y_d]$. Then the following hold*:

(a) *If R is quasi-unmixed, then A is locally quasi-unmixed.*

(b) *If P is a prime ideal in R_k such that $(M, f_0, \cdots, f_d)R_k\subseteq P$ and such that $f_0, \cdots, f_d$ are a subset of a system of parameters in R_{kP}, then PA is a prime ideal of A*, height $PA=$height $P-d$, *and* depth $PA=$depth $P+d$.

(c) *If R is quasi-unmixed and P is given in* (b) *then R is a quasi-subspace of A_Q, for all prime ideals Q in A such that $PA\subseteq Q$.*

PROOF. For (1), if R is quasi-unmixed, then the conclusion will follow from (2). The converse follows by Lemma 5. (2)(b) is proven in [**6**] and is stated here for convenience. By using Remark 4, (2)(a) can be proven by adapting the proof in [**6**]. By noting that the proof of the fact that R_{kP} is a dense subspace of R^*_{kP*} in [**6**] is valid if R is quasi-unmixed, (c) then follows by adapting the remainder of the proof in [**6**] and using the lemmas in this paper. Q.E.D.

COROLLARY 7 (CF. [**6**, COROLLARY 4.8]). *Let (R, M) be a local ring of* altitude $n\geqq 1$. *Assume that M is not a prime divisor of zero. Then the following are equivalent*:

(1) *R is quasi-unmixed.*

(2) *There exist analytically independent elements $x_0, x_1, \cdots, x_{n-1}$ in R such that x_0 is not a zero-divisor and such that R is a quasi-subspace of A_{MA}, where $A=R[x_1/x_0, \cdots, x_{n-1}/x_0]$.*

(3) *For every system of parameters* $x_0, \cdots, x_{n-1}$ *in* R *such that* x_0 *is not a zero-divisor,* R *is a quasi-subspace of* A_{MA}, *where* A *is given in* (2).

(4) *There exists a finitely generated ring* A *over* R *such that* $R \subseteq A \subseteq T$ *where* T *is the total quotient ring of* R, *and there exists a prime ideal* P *in* A *such that* R *is a quasi-subspace of* A_P *and* A_P *is quasi-unmixed.*

Let $B=(b_1, \cdots, b_k)R$ be an ideal in a Noetherian ring R. Let t be an indeterminant, and let $u=1/t$. The *Rees ring* $\mathscr{R}=\mathscr{R}(R, B)$ *of* R *with respect to* B is the ring $\mathscr{R}=R[u, tb_1, \cdots, tb_k]$. $\mathscr{R}$ is a graded Noetherian subring of $R[u, t]$. If (R, M) is a local ring, then $\mathscr{M}=(M, u, tb_1, \cdots, tb_k)$ is the unique maximal homogeneous ideal of $\mathscr{R}$ [7, Theorem 3.1, step (ii)]. By [6, Remark 3.10(ii)], if $b_1, \cdots, b_k$ form a system of parameters in the local ring (R, M), then $p=(M, u)\mathscr{R}$ is a height one depth k prime ideal in $\mathscr{R}$, and p is the radical of $u\mathscr{R}$ (and so p is the unique height one prime divisor of $u\mathscr{R}$).

The characterization of certain concepts of a ring R via the transition to a Rees ring has often been useful, and indeed this is the case here. Corollary 9 and the equivalence of (1) and (4) in Corollary 8 are the main results of this paper used in **[8]**.

COROLLARY 8 (CF. [**6**, COROLLARY 4.9]). *Let* (R, M) *be a local ring of* altitude $n \geqq 0$. *The following are equivalent:*

(1) R *is quasi-unmixed.*

(2) *There exist an ideal* B *in* R *and a prime ideal* P *of* $\mathscr{R}=\mathscr{R}(R, B)$ *such that* R *is a quasi-subspace of* $\mathscr{R}_P$ *and* $\mathscr{R}_P$ *is quasi-unmixed.*

(3) *There exists an ideal* B *in* R *such that* $\mathscr{R}_{\mathscr{M}}$ *is quasi-unmixed, where* $\mathscr{R}=\mathscr{R}(R, B)$ *and* $\mathscr{M}$ *is the maximal homogeneous ideal of* $\mathscr{R}$.

(4) *For every ideal* B *of* R *that is generated by a system of parameters,* R *is a quasi-subspace of* $\mathscr{R}_{(M,u)\mathscr{R}}$, *where* $\mathscr{R}=\mathscr{R}(R, B)$.

(4′) *There exists an ideal* B *of* R *that is generated by a system of parameters such that* R *is a quasi-subspace of* $\mathscr{R}_{(M,u)\mathscr{R}}$, *where* $\mathscr{R}=\mathscr{R}(R, B)$.

COROLLARY 9. *Let* (R, M) *be a local ring with completion* (R^*, M^*). *Let* B *be an* M*-primary ideal of* R *that is generated by a system of parameters. Let* $\mathscr{R}=\mathscr{R}(R, B)$ *and* $\mathscr{R}^*=\mathscr{R}(R^*, BR^*)$. *Then* R *is quasi-unmixed if and only if* $(M^*, u)\mathscr{R}^*$ *contains all minimal prime divisors of zero in* $\mathscr{R}^*$.

PROOF. Use Corollary 8 ((1) and (4′)) and Lemma 3. Q.E.D.

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