A CRITERION FOR THE POSITIVITY OF SINE POLYNOMIALS

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ABSTRACT. A sufficient condition is derived for a sine polynomial with real coefficients to be nonnegative on $(0, \pi)$.

We consider sine polynomials with real coefficients, and prove the following theorem.

THEOREM. *If*

$$(1) \quad F(x) = A_1 \sin x + A_2 \sin 2x + \cdots + A_n \sin nx,$$

and if

$$(2) \quad A_k \geq 2 \sum_{v=1}^{n-k} (-1)^{v+1} A_{k+v}, \quad k = 1, \cdots, n-1, \quad \text{and} \quad A_n > 0,$$

then

$$(3) \quad F(x) \geq 0 \quad \text{for } 0 < x < \pi.$$

Inequality (3) is strict on $(0, \pi)$ if and only if $(k_1, \cdots, k_s, n) \leq 2$, where $k_1, \cdots, k_s$ are those $k, 1 \leq k \leq n-1$, for which strict inequality holds in (2), and $(k_1, \cdots, k_s, n)$ is their greatest common divisor with n .

PROOF. For any $n \geq 1$, we have

$$(4) \quad \sigma_n(x) := \sum_{v=1}^n \sin(v - \tfrac{1}{2})x \geq 0, \quad 0 < x < 2\pi,$$

and therefore also

$$(5) \quad S_n(x) := \sum_{v=1}^n b_v \sin(v - \tfrac{1}{2})x \geq 0, \quad 0 < x < 2\pi,$$

if

$$(6) \quad b_1 \geq b_2 \geq \cdots \geq b_n > 0,$$

as a summation by parts will show:

$$(7) \quad S_n(x) = \sum_{k=1}^{n-1} (b_k - b_{k+1})\sigma_k(x) + b_n\sigma_n(x).$$

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Hence, if (6) is satisfied, we have

$$\sum_{v=1}^n b_v \sin(v - \tfrac{1}{2})x \cos \frac{x}{2} \geq 0, \quad 0 < x < \pi,$$

that is,

$$\sum_{v=1}^n b_v \{\sin(v-1)x + \sin vx\} \geq 0,$$

or again,

$$\sum_{v=1}^n A_v \sin vx \geq 0, \quad 0 < x < \pi,$$

where

$$(8) \quad A_v = b_v + b_{v+1} \quad \text{for } 1 \leq v \leq n-1, \text{ and } A_n = b_n.$$

Now if the A 's and the b 's are related as in (8), then conditions (2) and (6) are equivalent. Indeed, (2) can be written as

$$b_k + b_{k+1} \geq 2 \sum_{v=1}^{n-k-1} (-1)^{v+1} (b_{k+v} + b_{k+v+1}) + 2(-1)^{n-k+1} b_n = 2b_{k+1}$$

if $k \leq n-1$, and if $k=n$, $b_n = A_n > 0$. This completes the proof of (3).

The determination of cases of equality in (3) rests on their determination in (5); for this we require the familiar identity

$$(9) \quad \sigma_n(x) = \sin^2(nx/2)/\sin(x/2).$$

Let $k_1, \dots, k_s$ be those indices k for which $b_k > b_{k+1}$. It is obvious from (7) and (9) that the zeros of $S_n(x)$ are the common zeros of

$$\sin(k_1 x/2), \dots, \sin(k_s x/2), \sin(nx/2).$$

And the common zeros of these functions are at $x = 2m\pi/(k_1, \dots, k_s, n)$, $m \in \mathbb{Z}$, where $(k_1, \dots, k_s, n)$ denotes the greatest common divisor of $k_1, \dots, k_s, n$. Thus, these functions have a common zero on $(0, \pi)$ if and only if $(k_1, \dots, k_s, n) > 2$. This observation concerning (5), together with the relation between A 's and b 's, yields the statement concerning equality in (3). The proof of the Theorem is now complete.

COROLLARY. *Let F be defined as in (1). If $A_1 \geq A_2 \geq \dots \geq A_{n-1} \geq 2A_n > 0$, if $2A_k \leq A_{k-1} + A_{k+1}$ for $k=2, \dots, n-2$, and if $2A_{n-1} \leq A_{n-2} + 2A_n$, then $F(x) \geq 0$ for $0 < x < \pi$.*

This is a theorem of Fejér [4, Satz 27]; it is easily obtained as a corollary of the theorem we have just proved by verifying that Fejér's conditions (that the sequence $A_1, \dots, A_{n-1}, 2A_n$ be positive, nonincreasing and convex) imply conditions (2).

Examples show that Fejér's conditions are more restrictive than those in (2). For instance, the Theorem implies the positivity of $4 \sin x + 3 \sin 2x + (2-\varepsilon) \sin 3x + \frac{1}{2} \sin 4x$ for $0 < x < \pi$ and $0 \leq \varepsilon \leq \frac{1}{2}$, but the Corollary applies only to the case $\varepsilon=0$.

The argument used to prove the Theorem shows that

$$(10) \quad \sum_{v=1}^n a_v \sin(2v-1)x \geq 0, \quad 0 < x < \pi,$$

if and only if

$$\sum_{v=1}^n \alpha_v \sin vx \geq 0, \quad 0 < x < \pi,$$

where $\alpha_v = a_v + a_{v+1}$ for $v=1, \dots, n-1$, and $\alpha_n = a_n$.

On the other hand, a similar argument shows that (10) holds if and only if

$$(11) \quad \frac{1}{2} \lambda_0 + \sum_{v=1}^{n-1} \lambda_v \cos vx \geq 0, \quad 0 < x < 2\pi,$$

where $a_v = \lambda_{v-1} - \lambda_v$, $v=1, \dots, n-1$, and $a_n = \lambda_{n-1}$ (multiply the cosine polynomial in (11) by $\sin x/2$). This shows that the Theorem we have proved is equivalent to a theorem of Fejér [3, Theorem II] which states that (11) holds if the sequence $\lambda_0, \lambda_1, \dots, \lambda_n, 0, 0$ is convex and the λ 's are positive. In fact, one of Fejér's proofs of his theorem [5] is quite similar to the proof we have given of (3).

Turán has found another consequence of (10), namely

$$\sum_{v=1}^n \frac{a_v}{v} \geq 0, \quad 0 < x < \pi,$$

with strict inequality unless all a 's vanish ([1], [6], [7]).

These results can be placed in a more general setting by considering the Jacobi polynomials $P_n^{(\alpha, \beta)}$, defined by

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}],$$

$\alpha, \beta > -1$ [8]. Since

$$P_n^{(1/2, -1/2)}(\cos \theta) / P_n^{(-1/2, 1/2)}(1) = \sin(n + \frac{1}{2})\theta / \sin \frac{1}{2}\theta,$$

(4) is the same as

$$(12) \quad \sum_{k=0}^n \frac{P_k^{(\alpha, \beta)}(x)}{P_k^{(\beta, \alpha)}(1)} \geq 0, \quad -1 < x < 1,$$

for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$. R. Askey [2] has shown that (12) holds for all $n \geq 0$ if

$\alpha + \beta \geq 0$, $-1 < \alpha \leq \beta + 1$. Now if (12) holds, so does

$$(13) \quad \sum_{k=0}^n b_k \frac{P_k^{(\alpha, \beta)}(x)}{P_k^{(\beta, \alpha)}(1)} \geq 0, \quad -1 < x < 1,$$

if $b_0 \geq b_1 \geq \cdots \geq b_n > 0$ (sum by parts); for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, this is (5).

In this setting, our proof of (3) can be formulated thus: (3) follows from (13) and the recurrence relation

$$\frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\beta, \alpha)}(1)} + \frac{P_{n+1}^{(\alpha, \beta)}(x)}{P_{n+1}^{(\beta, \alpha)}(1)} = \frac{(2n + \alpha + \beta + 2)}{(2\beta + 1)} (1 + x) \frac{P_n^{(\alpha, \beta+1)}(x)}{P_n^{(\beta+1, \alpha)}(1)}$$

on setting $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, since

$$P_n^{(1/2, 1/2)}(\cos \theta) / P_n^{(1/2, 1/2)}(1) = \sin(n+1)\theta / (n+1)\sin \theta.$$

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