A FIXED POINT THEOREM FOR MULTIFUNCTIONS IN A LOCALLY CONVEX SPACE

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ABSTRACT. Let S be a convex subset of a locally convex space E and K a compact subset of S . Let $f: S \rightarrow F$ (a topological space) and $g: K \rightarrow F$ be multifunctions. In this paper sufficient conditions are given for the existence of an $x \in K$ such that $f(x) \cap g(x) \neq \emptyset$. The result generalizes a recent theorem of Himmelberg (J. Math. Anal. Appl. **38** (1972), 205–207) and in special cases extends some of the other well-known results.

The classical Tychonoff's fixed point theorem [7] has been extended to multifunctions by Browder [1], Fan [2], Glicksberg [4] and several others. Recently, Himmelberg [5] proved the following result, which is a generalization of [2].

THEOREM 1 (HIMMELBERG). *Let S be a nonvoid convex subset of a separated locally convex space E . Let $f: S \rightarrow S$ be an u.s.c. multifunction such that $f(x)$ is closed and convex for each $x \in S$, and $f(S)$ is contained in some compact set K of S . Then f has a fixed point.*

The purpose of this paper is to extend Theorem 1. Recall that a multifunction $f: X \rightarrow Y$ is a point to set function, assigning to each $x \in X$ a nonempty subset $f(x)$ of Y . If $A \subseteq X$, then $f(A) = \bigcup \{f(x) : x \in A\}$ and if $B \subseteq Y$, $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$. It is obvious that if $B = \bigcup B_i \subseteq Y$, then $f^{-1}(B) = \bigcup f^{-1}(B_i)$. If X and Y are topological spaces, the multifunction $f: X \rightarrow Y$ is upper semicontinuous iff for each closed set $A \subseteq Y$, $f^{-1}(A)$ is a closed subset of X . It is point closed (convex, compact) iff for each $x \in X$, $f(x)$ is a closed (convex, compact) subset of Y .

The proof of the following lemmas can be constructed directly from the definitions.

LEMMA 1. *Let $f: X \rightarrow Y$ be a point closed, u.s.c. multifunction on a topological space X into a regular space Y . Let $\{x_\alpha : \alpha \in \Gamma\}$ be a net in X which converges to an element $x \in X$. If $y_\alpha \in f(x)$ for each $\alpha \in \Gamma$ and if $y_\alpha \rightarrow y$, then $y \in f(x)$.*

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LEMMA 2. *Let X, Y be topological spaces. If $f: X \rightarrow Y$ is a point compact, u.s.c. multifunction, then the image $f(A)$ of a compact set $A \subseteq X$ is a compact subset of Y .*

The following result extends Theorem 1.

THEOREM 2. *Let S be a nonempty convex subset of a separated locally convex space E and K a compact subset of S . Let F be a regular separated topological space and $f: S \rightarrow F$ be point closed, u.s.c. and $g: K \rightarrow F$ be point compact, u.s.c. multifunctions. If for $x \in S$, (a) $f(x) \cap g(K) \neq \emptyset$, (b) $g^{-1}(f(x))$ is convex, then there is an $x \in K$ such that $f(x) \cap g(x) \neq \emptyset$.*

PROOF. Let $\mathcal{U}$ be a neighborhood basis of 0 (null element) consisting of a closed, convex and symmetric subset of E , and let U be a fixed element of $\mathcal{U}$. Since K is compact, there exists a finite subset $A = \{x_i: i=1, 2, \dots, n\} \subseteq K$ such that $K = \bigcup_{i=1}^n ((x_i + U) \cap K)$. It follows, therefore, that

$$(1) \quad f^{-1}(g(K)) \subseteq \bigcup_{i=1}^n f^{-1}(g((x_i + U) \cap K)).$$

Let $S(U)$ be the convex hull of the set A . Since A is finite, $S(U)$ is a compact subset of E , and by hypothesis (a) and (1) we have

$$(2) \quad S(U) \subseteq \bigcup_{i=1}^n f^{-1}(g((x_i + U) \cap K)).$$

Define a multifunction $f_U: S(U) \rightarrow S(U)$ by

$$(3) \quad f_U(x) = (g^{-1}(f(x)) + U) \cap S(U).$$

Note that if $x \in S(U)$, then, by (2), $x_i \in g^{-1}(f(x)) + U$ for some $i=1, 2, \dots, n$ and hence $f_U(x) \neq \emptyset$. We show that f_U is a point convex, closed u.s.c. multifunction in the relative topology of $S(U)$. The convexity of $f_U(x)$, $x \in S(U)$, is obvious by (3). Also, since $f(x)$ is closed, the u.s.c. of g implies that $g^{-1}(f(x))$ is closed and hence compact in K . Therefore, $g^{-1}(f(x)) + U$ is closed in E , that is $f_U(x)$ is closed in $S(U)$. To show the u.s.c. of f_U , let A be a closed (hence compact) set in $S(U)$. Then

$$\begin{aligned} f_U^{-1}(A) &= \{x \in S(U): (g^{-1}(f(x)) + U) \cap S(U) \cap A \neq \emptyset\} \\ (4) \quad &= \{x \in S(U): x \in f^{-1}(g((A + U) \cap K))\} \\ &= f^{-1}(g((A + U) \cap K)) \cap S(U). \end{aligned}$$

Now, since $A + U$ is closed in E , it follows by Lemma 2 that $g((A + U) \cap K)$ is compact and therefore $f^{-1}(g((A + U) \cap K))$ is closed in S . If

$$f^{-1}(g((A + U) \cap K)) = C \cap S$$

where C is some closed set in E , then (4) implies that $f_U^{-1}(A) = C \cap S(U)$ which is a closed set in $S(U)$. Thus, f_U is an u.s.c. multifunction. Now $S(U)$ with its relative topology being homeomorphic to the Euclidean n -ball (with the usual topology), it follows by Kakutani's theorem [6] that for each $U \in \mathcal{U}$, there exists an element $x_U \in S(U)$ such that $x_U \in f_U(x_U)$, that is

$$g((x_U + U) \cap K) \cap f(x_U) \neq \emptyset.$$

Let $z_U \in U$ and $y_U \in g(K)$ such that

$$(5) \quad y_U \in g(x_U + z_U) \cap f(x_U).$$

Then the net $\{z_U: U \in \mathcal{U}\} \rightarrow 0$. Also, since $x_U + z_U \in K$, there is a subnet $\{x_{U'} + z_{U'}: U' \in \mathcal{U}' \subseteq \mathcal{U}\} \rightarrow x \in K$. Now $y_{U'} \in g(K)$ implies that a subnet $\{y_{U''}: U'' \in \mathcal{U}'' \subseteq \mathcal{U}'\} \rightarrow y \in g(K)$. Therefore, restricting to the directed set $\mathcal{U}''$, it follows by Lemma 1 and (5) that $y \in g(x) \cap f(x)$. This establishes the proof of Theorem 2.

COROLLARY 1. *Let S be a nonempty convex subset of a separated locally convex space E , and K a compact subset of S . Let $f: S \rightarrow E$ be point closed, u.s.c. and $g: K \rightarrow E$ be point compact, u.s.c. multifunctions. If for $x \in S$, (a) $f(x) \cap g(K) \neq \emptyset$, (b) $g^{-1}(f(x))$ is convex, then there is an $x \in K$ such that $f(x) \cap g(x) \neq \emptyset$.*

If the space $F \equiv E$ in Theorem 2 and g therein is the identity mapping of K onto itself, then we have

COROLLARY 2. *Let S be a nonempty convex subset of a separated locally convex space E and K a compact subset of S . Let $f: S \rightarrow E$ be a point closed, u.s.c. multifunction such that $f(x) \cap K$ is a nonempty convex set for each $x \in S$. Then f has a fixed point in K .*

It may be remarked that, when $S = K$, Corollary 1 provides a slightly weaker form of a result of Browder's theorem 4 [1]. Further, Theorem 2 is a generalization of Fan's theorem 2 [3] if the space Y therein is locally convex.

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