INJECTIVE DIMENSION OF QUATERNION ORDERS

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ABSTRACT. Tarsy has shown that if R is a discrete valuation ring with quotient field K , and Σ is a quaternion K -algebra, then the finitistic global dimension of any R -order in Σ is 1. In this paper we allow R to be any regular local ring of dimension n and study the R -free orders Λ in Σ . First we show that the finitistic global dimension of Λ is n . Our main result concerns the injective dimension of Λ (considered as either a left or a right Λ -module). Let $\mathfrak{M}$ denote the maximal ideal of R . Then the injective dimension of Λ is n , unless $\Lambda/\mathfrak{M}\Lambda$ is a commutative local ring whose socle is not principal. In this case, the injective dimension of Λ is ∞ .

Let $(R, \mathfrak{M})$ be a regular local ring with quotient field K and let Σ be a quaternion K -algebra (i.e. a central simple K -algebra with $[\Sigma:K]=4$). Tarsy [5, Theorem 7] has proved that when R is a discrete valuation ring (i.e. $\text{gl dim } R=1$) the finitistic global dimension (f gl dim) of any R -order in Σ is 1. The hypothesis that $[\Sigma:K]=4$ is crucial, since Tarsy exhibits an R -order in $M_q(K)$, the ring of $q \times q$ matrices over K , whose global dimension is $q-1$.

A fairly trivial generalization of Tarsy's theorem is

PROPOSITION 3. *Suppose $\text{gl dim } R=n$ and Λ is an R -free order in Σ . Then $\text{f gl dim } \Lambda=n$.*

The main purpose of this note, however, is to extend Tarsy's proof to a determination of the self-injective dimension of an R -free order in Σ .

THEOREM 5. *Suppose $\text{gl dim } R=n$ and Λ is an R -free order in Σ . Then $\text{inj dim}_\Lambda \Lambda=n$ if $\Lambda/\mathfrak{M}\Lambda$ is not a commutative local ring. If $\Lambda/\mathfrak{M}\Lambda$ is a commutative local ring, then $\text{inj dim}_\Lambda \Lambda=n$ or ∞ according as the socle of $\Lambda/\mathfrak{M}\Lambda$ is, or is not, principal.*

Various examples are given, including one of an order Λ which *does* have infinite self-injective dimension.

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LEMMA 1. *Let $(R, \mathfrak{M})$ be a regular local ring of dimension at least 2. Let Λ be an R -free order in Σ . Then there exists an x in $\mathfrak{M} - \mathfrak{M}^2$ such that $\Lambda_{(x)}/x\Lambda_{(x)}$ is a quaternion algebra over $R_{(x)}/xR_{(x)}$ and $\Lambda/x\Lambda$ is an R/xR -free order in $\Lambda_{(x)}/x\Lambda_{(x)}$.*

PROOF. Let $h(\Lambda/R)$ be the homological different of Λ over R (see [1, p. 378]). Now $h(\Lambda/R) \otimes_R K = h(\Sigma/K)$ and the latter is nonzero because Σ is a central separable K -algebra. So $h(\Lambda/R)$ is a nonzero ideal of R , and is thus contained in only finitely many minimal nonzero prime ideals of R . There are infinitely many distinct prime ideals of the form (z) , z in $\mathfrak{M} - \mathfrak{M}^2$. Hence there is one, say (x) , which does not contain $h(\Lambda/R)$. Therefore $\Lambda_{(x)}$ is central separable over $R_{(x)}$ and so $\Lambda_{(x)}/x\Lambda_{(x)}$ is central separable (=central simple) over $R_{(x)}/xR_{(x)}$. Since Λ is R -free, $\Lambda/x\Lambda$ is R/xR -free and is easily seen to be an R/xR -order in $\Lambda_{(x)}/x\Lambda_{(x)}$. Finally, $4 = [\Sigma:K] = \text{rank}_R \Lambda = \text{rank}_{R/xR} \Lambda/x\Lambda = [\Lambda_{(x)}/x\Lambda_{(x)}:R_{(x)}/xR_{(x)}]$. ■

LEMMA 2. *Let S be a left noetherian ring and let x be a central nonzero divisor contained in $\text{Rad } S$, the Jacobson radical of S . Then $\text{left f gl dim } S = 1 + \text{left f gl dim } S/xS$.*

PROOF. Let A be a left S/xS -module of finite projective dimension (p.d.) Then $\text{p.d.}_S A = 1 + \text{p.d.}_{S/xS} A$ [3, Theorem 5.3]. Hence $\text{left f gl dim } S \geq 1 + \text{left f gl dim } S/xS$.

For the reverse inequality, suppose M is a left S -module with $\text{p.d.}_S M = t$. Let $0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0$ be exact with F S -free. Then $\text{p.d.}_S L = t - 1$ and x is regular on L . Thus by [3, Theorem 5.6] we have $\text{p.d.}_{S/xS} L/xL = t - 1$, so $t - 1 \leq \text{left f gl dim } S/xS$, i.e. $t \leq 1 + \text{left f gl dim } S/xS$. Therefore $\text{left f gl dim } S \leq 1 + \text{left f gl dim } S/xS$, and so the equality is established. ■

PROPOSITION 3. *Suppose $\text{gl dim } R = n$ and Λ is an R -free order in Σ . Then $\text{f gl dim } \Lambda = n$.*

PROOF. By induction on n . The case $n = 1$ is Tarsy's result [5, Theorem 7]. Now assume $n > 1$ and the theorem is true for $n - 1$. By Lemma 1, for some $x \in \mathfrak{M} - \mathfrak{M}^2$, $\Lambda/x\Lambda$ is an R/xR -free order in the quaternion algebra $\Lambda_{(x)}/x\Lambda_{(x)}$. Since $\text{gl dim } R/xR = n - 1$, by the induction hypothesis $\text{f gl dim } \Lambda/x\Lambda = n - 1$. The desired result now follows from Lemma 2. ■

LEMMA 4. *Let F be a field and S an F -algebra such that $[S:F] = 4$, S is quasi-local, and S is noncommutative. Suppose that for every $z \in S - F$, $[F[z]:F] = 2$. Then S is a Frobenius algebra. In particular, S is quasi-Frobenius, i.e. $\text{inj dim}_S S = 0$.*

PROOF. By hypothesis, $S/\text{Rad } S$ is a simple artin ring of dimension at most 4 over F . If S is not local, i.e. if $S/\text{Rad } S$ is not a division ring, then

$S/\text{Rad } S \approx M_2(F)$, the ring of 2×2 matrices over F . Hence $[S/\text{Rad } S:F] = 4 = [S:F]$ and so $\text{Rad } S = 0$. But then $S \approx M_2(F)$ which is Frobenius since the trace map yields an associative, nondegenerate bilinear form $M_2(F) \times M_2(F) \rightarrow F$, namely $(x, y) \mapsto \text{tr}(xy)$.

We may assume, therefore, that $\text{Rad } S \neq 0$ and that S is local, so that $\text{Rad } S$ is the unique maximal left ideal and the unique maximal right ideal of S . Thus if z is a nonunit, then $z \in \text{Rad } S$. Since S is artin, $\text{Rad } S$ is nilpotent. Hence z is nilpotent and since $[F[z]:F] = 2$, $z^2 = 0$.

Suppose that $\text{Rad } S$, as a left ideal, is principal, say $\text{Rad } S = Sx$. Then $xS \subset Sx$ and so $(\text{Rad } S)^2 = (Sx)(Sx) = S(xS)x \subset Sx^2 = 0$. Therefore $\text{Rad } S$ is the unique nonzero left (right) ideal of S . But then $\text{left ann}(\text{Rad } S) = \text{right ann}(\text{Rad } S) = \text{Rad } S$ and $[\text{ann}(\text{Rad } S):F] + [\text{Rad } S:F] = 2 + 2 = [S:F]$. Thus by [2, p. 414, Theorem 6.13], S is Frobenius.

Let us now assume that $\text{Rad } S$ is *not* principal. Clearly, then, $[\text{Rad } S:F] > 1$. We claim that $[\text{Rad } S:F] = 3$. For suppose $[\text{Rad } S:F] = 2$. Let $x \in \text{Rad } S$, $x \neq 0$. Since $Sx \subseteq \text{Rad } S$, $[Sx:F] = 1$, and thus for all $s \in S$, sx and x are F -dependent. Hence $sx = ax$ for some $a \in F$. Thus $(s-a)x = 0$, so $s-a$ is a nonunit and hence belongs to $\text{Rad } S$. Therefore $S = F + \text{Rad } S$, which implies that $[S:F] = 3$, a contradiction. Thus $[\text{Rad } S:F] = 3$, and since $\text{Rad } S \subseteq F + \text{Rad } S$, it follows that $F + \text{Rad } S = S$. We claim that $\text{Rad } S$ has an F -basis of the form $\{x, y, xy\}$. Suppose not. Then for every pair of F -independent elements x and y in $\text{Rad } S$, $xy = ax + by$ for some $a, b \in F$. Since $z^2 = 0$ for every $z \in \text{Rad } S$, left multiplication by x yields $0 = bxy$, and right multiplication by y yields $0 = axy$. If $xy \neq 0$, then $a = b = 0$. But then $xy = 0 \cdot x + 0 \cdot y = 0$. Thus the product of any two F -independent elements of $\text{Rad } S$ is 0. On the other hand, if $z, w \in \text{Rad } S$ and $w = az$ for some $a \in F$, then $zw = az^2 = 0$. Thus $(\text{Rad } S)^2 = 0$, and so $\text{Rad } S$ is commutative. Hence the centralizer of $\text{Rad } S$ in S contains $F + \text{Rad } S = S$. So $\text{Rad } S \subset \text{center of } S$. Thus $S = F + \text{Rad } S \subset \text{center of } S$, so S is commutative, which is a contradiction. Therefore $\text{Rad } S$ *must* have an F -basis $\{x, y, xy\}$. Hence $\{1, x, y, xy\}$ is an F -basis for S . Now $0 = (x+y)^2 = x^2 + y^2 + xy + yx = xy + yx$. So $yx = -xy$. Thus $S = F[1, x, y, xy]$, with the defining relations $x^2 = y^2 = 0$, and $yx = -xy$. Since S is not commutative, $yx \neq xy$, and so $\text{char}(F) \neq 2$.

It is easy to see that every left ideal and every right ideal of S is two-sided. Furthermore, the proper ideals are precisely $\text{Rad } S = (x, y)$, $(\text{Rad } S)^2 = (xy)$ (the unique minimal ideal), and $(ax + by)$, $a, b \in F$. A simple calculation shows that each ideal I is the left (and right) annihilator of some ideal J . Specifically, $\text{ann}(x, y) = (xy)$, $\text{ann}(xy) = (x, y)$, and for all $a, b \in F$, $\text{ann}(ax + by) = (ax + by)$. Furthermore, we see that for each ideal I , $[\text{ann}(I):F] + [I:F] = 4 = [S:F]$. Hence by [2, Theorem 6.13], S is Frobenius. ■

REMARK. None of the four hypotheses of Lemma 4 can be dropped. We show this with four examples.

(i) $S = F[X, Y, Z]/(X, Y, Z)^2$, X, Y, Z independent indeterminates.

S is a commutative local ring, $[S:F]=4$, and for every $w \in S-F$, $[F[w]:F]=2$. $\text{Socle}(S)=(\bar{X}, \bar{Y}, \bar{Z})$ is not principal so $\text{inj dim}_S S \neq 0$. (Since S is a local artin ring, $\text{inj dim}_S S = \infty$.)

$$(ii) \quad S = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & a \end{pmatrix} \mid a, b, c, d \in F \right\} = F[1, e_{21}, e_{31}, e_{32}].$$

Clearly S is noncommutative and $[S:F]=4$.

$$\text{Rad } S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} \mid b, c, d \in F \right\}$$

and $S/\text{Rad } S \approx F$, so that S is local. Let

$$w = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = e_{21} + e_{32}.$$

Then $w^2 = e_{31} \neq 0$, but $w^3 = 0$. So $[F[w]:F]=3$.

If $\text{inj dim}_S S = 0$, then for all left ideals I , $\text{l. ann}(\text{r. ann}(I)) = I$. Now let $I = Se_{31}$. Then $\text{r. ann}(I) = (e_{31}, e_{32})S = \text{Rad } S$, and $\text{l. ann}(\text{Rad } S) = Se_{21}$. Hence $I \not\subseteq \text{l. ann}(\text{r. ann}(I))$, and so $\text{inj dim}_S S \neq 0$.

$$(iii) \quad S = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & d \end{pmatrix} \mid a, b, c, d \in F \right\} = F[1, e_{11}, e_{21}, e_{31}].$$

Again, S is noncommutative and $[S:F]=4$. However $S/\text{Rad } S \approx F \oplus F$, so S is not quasi-local. If

$$z = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & d \end{pmatrix},$$

then $z^2 - (a+d)z + ad = 0$, so if $z \notin F$, $[F[z]:F]=2$.

Now $\text{r. ann}(Se_{21}) = S(e_{11}, e_{21}, e_{31}) = \text{r. ann}(Se_{31})$, although $Se_{21} \neq Se_{31}$.

Hence for at least one of these ideals $I, I \neq 1$. $\text{ann}(\text{r. ann}(I))$. Thus

$$\text{inj dim}_S S \neq 0.$$

$$(iv) \quad S = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & f & a & 0 \\ c & d & 0 & a \end{pmatrix} \mid a, b, c, d, f \in F \right\} = F[1, e_{31}, e_{41}, e_{32}, e_{42}].$$

Note that $[S:F]=5$. S is noncommutative and local. $(\text{Rad } S)^2=0$. Hence if $z \in S$ and z has a 's on its main diagonal, then $z-a \in \text{Rad } S$, so $(z-a)^2=0$. Hence $[F[z]:F] \leq 2$. Since $(\text{Rad } S)^2=0$, if $w \in \text{Rad } S$ then $\text{r. ann}(Sw)=\text{Rad } S=\text{r. ann}(\text{Rad } S)$. Since $Se_{31} \subseteq \text{Rad } S$, $\text{inj dim}_S S \neq 0$.

We come now to the main result of this paper.

THEOREM 5. *Suppose $\text{gl dim } R=n$ and Λ is an R -free order in the quaternion K -algebra Σ . Then $\text{inj dim}_\Lambda \Lambda=n$ if $\Lambda/\mathfrak{M}\Lambda$ is not a commutative local ring. If it is, then $\text{inj dim}_\Lambda \Lambda$ is either n or ∞ , according as the socle of $\Lambda/\mathfrak{M}\Lambda$ (i.e. $\text{ann}(\text{Rad } \Lambda/\mathfrak{M}\Lambda)$) is, or is not, principal.*

PROOF. By induction on n . Let $n=1$. Passage to the completion $(\hat{R}, \hat{\mathfrak{M}})$ of the local ring $(R, \mathfrak{M})$ alters none of the hypotheses. Furthermore, if $\mathfrak{M}=(x)$, then $\text{inj dim}_\Lambda \Lambda=1+\text{inj dim}_{\Lambda/x\Lambda} \Lambda/x\Lambda=\text{inj dim}_{\hat{\Lambda}} \hat{\Lambda}$. The second equality holds because $\Lambda/x\Lambda=\Lambda/\mathfrak{M}\Lambda \approx \hat{\Lambda}/\mathfrak{M}\hat{\Lambda}=\hat{\Lambda}/x\hat{\Lambda}$. So assume now that R is complete. Then if Λ has more than one maximal two-sided ideal, by Tarsy's proof of [5, Theorem 7] $\Lambda/x\Lambda \approx \Gamma/x\Gamma$ where $\Gamma=(\begin{smallmatrix} R & (x) \\ R & R \end{smallmatrix})$. But $\text{gl dim } \Gamma=1$, so $\text{inj dim}_\Gamma \Gamma=1$. Hence $\text{inj dim}_{\Lambda/x\Lambda} \Lambda/x\Lambda=\text{inj dim}_{\Gamma/x\Gamma} \Gamma/x\Gamma=0$, and so $\text{inj dim}_\Lambda \Lambda=1$.

So assume that Λ has just one maximal two-sided ideal. Thus $\Lambda/x\Lambda=\Lambda/\mathfrak{M}\Lambda$ is quasi-local and $[\Lambda/\mathfrak{M}\Lambda:R/\mathfrak{M}]=\text{rank}_R \Lambda=4$. If $z \in \Lambda-R$ then $[K[z]:K]=2$ since Σ is a quaternion algebra. Since R is a discrete valuation ring and $R[z] \subset \Lambda$, and Λ is R -free, it follows that $R[z]$ is R -free of rank 2. Hence for any $\bar{z} \in \Lambda/\mathfrak{M}\Lambda-R/\mathfrak{M}$, $[R/\mathfrak{M}[\bar{z}]:R/\mathfrak{M}]=2$. Thus by Lemma 4, if $\Lambda/\mathfrak{M}\Lambda$ is not commutative, $\text{inj dim}_{\Lambda/\mathfrak{M}\Lambda} \Lambda/\mathfrak{M}\Lambda=0$. Therefore $\text{inj dim}_\Lambda \Lambda=1$.

On the other hand, if $\Gamma=\Lambda/\mathfrak{M}\Lambda$ is commutative and local, then Γ is self-injective if and only if $\text{ann}(\text{Rad } \Gamma)=\text{socle}(\Gamma)$ is principal. Otherwise $\text{inj dim}_\Gamma \Gamma=\infty$. Hence $\text{inj dim}_\Lambda \Lambda=1$ or ∞ , according as $\text{socle}(\Lambda/\mathfrak{M}\Lambda)$ is, or is not, principal.

Now assume that $n>1$ and that the theorem is true for $n-1$. By Lemma 1, for some $x \in \mathfrak{M}-\mathfrak{M}^2$, $\Lambda/x\Lambda$ is an R/xR -free order in a quaternion algebra. $\text{gl dim } R/xR=n-1$ so if $\Lambda/\mathfrak{M}\Lambda$ is not commutative and local, by our induction hypothesis $\text{inj dim}_{\Lambda/x\Lambda} \Lambda/x\Lambda=n-1$, and thus $\text{inj dim}_\Lambda \Lambda=n$. If $\Lambda/\mathfrak{M}\Lambda$ is commutative and local then $\text{inj dim}_{\Lambda/x\Lambda} \Lambda/x\Lambda=n-1$ or ∞

according as $\text{socle}(\Lambda/\mathfrak{M}\Lambda)$ is, or is not, principal. Thus $\text{inj dim}_\Lambda \Lambda = n$ or ∞ according as $\text{socle}(\Lambda/\mathfrak{M}\Lambda)$ is, or is not, principal. ■

We conclude this paper with 3 examples which show that each of the situations described in Theorem 5 *does* in fact occur.

EXAMPLE A. Let $(R, \mathfrak{M})$ be a discrete valuation ring and let $\Lambda = M_2(R)$. Then $\Lambda/\mathfrak{M}\Lambda = M_2(R/\mathfrak{M})$, which is noncommutative and quasi-local but not local. $\text{inj dim}_\Lambda \Lambda = \text{gl dim } \Lambda = \text{gl dim } R = 1$.

EXAMPLE B. Let $R = \mathbb{Z}/2\mathbb{Z}[[t]]$, the ring of formal power series in one variable over $\mathbb{Z}/2\mathbb{Z}$. Let $\Lambda = R[1, \alpha, \beta, \alpha\beta] \subset M_2(R)$ where

$$\alpha = \begin{pmatrix} 1+t & 1 \\ 1+t+t^2 & 1+t \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 1 \\ 1+t & 1 \end{pmatrix}.$$

If we identify each $r \in R$ with $r \cdot I$, where I denotes the 2×2 identity matrix, we obtain the relations $\alpha^2 = \beta^2 = t$, and $\alpha\beta = \beta\alpha + t^2$. $1, \alpha, \beta$, and $\alpha\beta$ are linearly independent, so Λ is R -free of rank 4 and Λ is an R -order in $M_2(K)$, where K is the quotient field of R .

$\Lambda/t\Lambda = \mathbb{Z}/2\mathbb{Z}[1, \bar{\alpha}, \bar{\beta}, \bar{\alpha}\bar{\beta}]$ where $\bar{\alpha}^2 = \bar{\beta}^2 = 0$ and $\bar{\alpha}\bar{\beta} = \bar{\beta}\bar{\alpha}$. Hence $\Lambda/t\Lambda \approx \mathbb{Z}/2\mathbb{Z}[X, Y]/(X^2, Y^2)$, which is a commutative local ring. $\text{Socle}(\Lambda/t\Lambda) = \Lambda/t\Lambda(\bar{\alpha}\bar{\beta})$, so $\Lambda/t\Lambda$ is self-injective. Hence $\text{inj dim}_\Lambda \Lambda = 1$.

EXAMPLE C. Let R be a discrete valuation ring with maximal ideal $\mathfrak{M} = (t)$. Let $\Lambda = \{ \begin{pmatrix} a & bt \\ ct & d \end{pmatrix} \in M_2(R) \mid a \equiv d \pmod{(t)} \}$. In other words, $\Lambda = R[1, te_{11}, te_{12}, te_{21}]$.

First we show that Λ is local. Map Λ onto $R/\mathfrak{M}$ by $\begin{pmatrix} a & bt \\ ct & d \end{pmatrix} \mapsto a + \mathfrak{M}$. The kernel of this ring homomorphism is $\{ \begin{pmatrix} a & bt \\ ct & d \end{pmatrix} \mid a, d \in (t) \} = I$ which is contained in $\text{Rad } \Lambda$. Thus $\Lambda/\text{Rad } \Lambda$ is a homomorphic image of $\Lambda/I \approx R/\mathfrak{M}$. Hence $I = \text{Rad } \Lambda$ and Λ is local. Next we check that $\Lambda/t\Lambda$ is commutative. Let $\alpha = te_{11}$, $\beta = te_{12}$, $\gamma = te_{21}$. Then $\alpha^2 = t\alpha$, $\beta^2 = \gamma^2 = 0$, $\alpha\beta = t\beta$, $\gamma\alpha = t\gamma$, and $\alpha\gamma = \beta\alpha = 0$. Also, $\beta\gamma = t\alpha$ and $\gamma\beta = t(t - \alpha)$. $\Lambda/t\Lambda = R/(t)[1, \bar{\alpha}, \bar{\beta}, \bar{\gamma}]$. From the above relations it follows that $\text{Rad}(\Lambda/t\Lambda) = \Lambda/t\Lambda(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $(\text{Rad } \Lambda/t\Lambda)^2 = 0$. Thus $\Lambda/t\Lambda$ is commutative and $\text{socle}(\Lambda/t\Lambda) = \text{Rad}(\Lambda/t\Lambda)$, which is not principal. So by the preceding theorem, $\text{inj dim}_\Lambda \Lambda = \infty$.

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