SPLITTING FIELDS AND SEPARABILITY

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ABSTRACT. It is a classical result that if $(R, \mathfrak{M})$ is a complete discrete valuation ring with quotient field K , and if $R/\mathfrak{M}$ is perfect, then any finite dimensional central simple K -algebra Σ can be split by a field L which is an unramified extension of K . Here we prove that if $(R, \mathfrak{M})$ is *any* regular local ring, and if Σ contains an R -order Λ whose global dimension is finite and such that $\Lambda/\text{Rad } \Lambda$ is central simple over $R/\mathfrak{M}$, then the existence of an “ R -unramified” splitting field L for Σ implies that Λ is R -separable. Using this theorem we construct an example which shows that if R is a regular local ring of dimension greater than one, and if its characteristic is not 2, then there is a central division algebra over K which has no R -unramified splitting field.

In the study of central simple algebras, splitting fields play an important role. A well-known classical result states that if Σ is a finite dimensional central simple algebra over a field K , then Σ can be split by a field L which is a finite separable extension of K . If K is the quotient field of a complete discrete valuation ring R whose residue class field $R/\mathfrak{M}$ is perfect, then L can be chosen so that the integral closure, S , of R in L , is R -separable [4, Chapter 5, Theorem 8]. Using this fact, Auslander and Goldman [2, Proposition 8.4] proved that if R is a discrete valuation ring (not necessarily complete) with perfect residue class field and if Λ is a maximal R -order in Σ such that the center of Λ/N is $R/\mathfrak{M}$ (where N is the Jacobson radical of Λ), then Λ is R -separable. Our purpose in this paper is to generalize this result, and thereby produce an example of a division algebra Σ which cannot be split by any “ R -unramified” extension L of K , where R is a regular local ring of dimension greater than one.

DEFINITION. Let R be a domain with quotient field K . A finite field extension L of K is *R -unramified* if L is the quotient field of some R -algebra S which is R -separable and finitely generated as an R -module.

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THEOREM. *Let $(R, \mathfrak{M})$ be a regular local ring with quotient field K . Let Σ be a finite dimensional central simple K -algebra which contains an R -order Λ such that $\text{gl dim } \Lambda < \infty$, and $\Lambda/\text{Rad } \Lambda$ is central simple over $R/\mathfrak{M}$. Suppose Σ has an R -unramified splitting field L . Then Λ is R -separable.*

PROOF. Assume first that R is complete. By hypothesis, L is the quotient field of S , an R -separable domain which is a finitely generated R -module. According to [1, Proposition 4.3], S is R -free. Since R is complete, so is S , and therefore any proper idempotent in $S/\text{Rad } S$ could be lifted to one in S . But S is a domain, and hence contains no proper idempotent, so $S/\text{Rad } S$ has none either and is therefore a field. Since S is R -separable, $\text{Rad } S = \mathfrak{M}S$. Hence S is local, with maximal ideal $\mathfrak{M}S$. It follows that S is a regular local ring.

Since $S/\mathfrak{M}S$ is a field, $S/\mathfrak{M}S \otimes_{R/\mathfrak{M}} \Lambda/N$ is central simple over $S/\mathfrak{M}S$, where $N = \text{Rad } \Lambda$. But $S/\mathfrak{M}S \otimes_{R/\mathfrak{M}} \Lambda/N \approx S \otimes_R \Lambda/N$, so the latter, in particular, is a simple ring. Since S is R -free, the sequence

$$0 \rightarrow S \otimes_R N \rightarrow S \otimes_R \Lambda \rightarrow S \otimes_R \Lambda/N \rightarrow 0$$

is exact. Hence $S \otimes_R N$ is a maximal two-sided ideal in $S \otimes_R \Lambda$. Furthermore, since $N = \text{Rad } \Lambda$ and $S \otimes_R N$ is a two-sided ideal, $S \otimes_R N \subset \text{Rad}(S \otimes_R \Lambda)$. Thus $S \otimes_R N = \text{Rad}(S \otimes_R \Lambda)$ and $S \otimes_R \Lambda$ is quasi-local (i.e. modulo its radical it is a simple ring). By [2, Theorem 1.8], $\text{gl dim } S \otimes_R \Lambda \leq \text{gl dim } \Lambda < \infty$. Thus $S \otimes_R \Lambda$ has finite global dimension and is a quasi-local S -order in $L \otimes_K \Sigma$, a full matrix ring over L . We have recently proved [3, Theorem 4] that under these conditions $S \otimes_R \Lambda$ is a full matrix ring over S and is therefore S -separable. Since S is R -separable, it follows from [2, Theorem 2.3] that $S \otimes_R \Lambda$ is R -separable. But S is R -free, so by [2, Proposition 1.7], Λ is R -separable.

Now, for the general case, let $(R, \mathfrak{M})$ be any regular local ring and let $(\hat{R}, \hat{\mathfrak{M}})$ be its completion in the $\mathfrak{M}$ -adic topology. Let $\hat{K}$ be the quotient field of $\hat{R}$, let $\hat{\Lambda} = \Lambda \otimes_R \hat{R}$, and let $\hat{\Sigma} = \Sigma \otimes_K \hat{K}$. Finally, let $\hat{L} = L \otimes_K \hat{K}$ and $\hat{S} = S \otimes_R \hat{R}$. Since L is K -separable, $\hat{L}$ is $\hat{K}$ -separable, and therefore $\hat{L} = L_1 \oplus \cdots \oplus L_s$ (ring direct product) where each L_i is a separable field extension of $\hat{K}$. If S_i denotes the image of $\hat{S}$ in L_i under the projection map $\hat{L} \rightarrow L_i$, then $\hat{S} = S_1 \oplus \cdots \oplus S_s$. Thus S_i is an $\hat{R}$ -separable domain with quotient field L_i . Since L splits Σ , $\hat{L}$ splits $\hat{\Sigma}$. Hence for each i , L_i splits $\hat{\Sigma}$. If $\hat{N} = N \otimes_R \hat{R}$, then $\hat{N} = \text{Rad } \hat{\Lambda}$, and $\hat{\Lambda}/\hat{N} \simeq \Lambda/N$. Hence $\hat{\Lambda}/\hat{N}$ is central simple over $\hat{R}/\hat{\mathfrak{M}}$. Also, $\text{gl dim } \hat{\Lambda} = \text{gl dim } \Lambda$. We may conclude, by our proof for the complete case, that $\hat{\Lambda}$ is $\hat{R}$ -separable. But this implies that Λ is R -separable.

Notation. For a commutative ring A , $B(A)$ will denote the Brauer group of A (cf. [2]).

COROLLARY. *Let $(R, \mathfrak{M})$ be a regular local ring with quotient field K and assume that $B(R/\mathfrak{M})=0$ (e.g. $R/\mathfrak{M}$ might be algebraically closed). Let $\Lambda \neq R$ be a central R -algebra, finitely generated as an R -module. Suppose that Λ is local and $\text{gl dim } \Lambda < \infty$. Then $\Lambda \otimes_R K$ is a central division algebra over K which has no R -unramified splitting field. (If $R/\mathfrak{M}$ is perfect, then the dimension of R is greater than one.)*

PROOF. $\Lambda \otimes_R K$ is an artin ring containing a local order with finite global dimension. By [3, Corollary 2], $\Lambda \otimes_R K$ is a division ring. Now if $\Lambda \otimes_R K$ has an R -unramified splitting field, then by the preceding theorem $\hat{\Lambda}$ is $\hat{R}$ -separable. Since $\hat{R}$ is complete, the map $B(\hat{R}) \rightarrow B(R/\mathfrak{M})$, induced by the projection map $\hat{R} \rightarrow \hat{R}/\hat{\mathfrak{M}} = R/\mathfrak{M}$, is a monomorphism [2, Corollary 6.2]. Since $B(R/\mathfrak{M})=0$, it follows that $B(\hat{R})=0$. Hence for some integer q , $\hat{\Lambda} \simeq M_q(\hat{R})$, the ring of $q \times q$ matrices over $\hat{R}$. Therefore $\Lambda/\text{Rad } \Lambda \simeq \hat{\Lambda}/\text{Rad } \hat{\Lambda} \simeq M_q(R/\mathfrak{M})$. But since Λ is local, $\Lambda/\text{Rad } \Lambda$ is a division ring. Hence $\Lambda/\text{Rad } \Lambda \simeq R/\mathfrak{M}$ and $q=1$. Thus $\hat{\Lambda} = \hat{R}$ and so $\Lambda = R$. Contradiction. Thus $\Lambda \otimes_R K$ has no R -unramified splitting field.

Since $\hat{R}$ and $\hat{\Lambda}$ share all the properties of R and Λ , $\hat{\Lambda} \otimes_{\hat{R}} \hat{K}$ has no $\hat{R}$ -unramified splitting field. Thus if $R/\mathfrak{M}$ is perfect, $\hat{R}$ cannot be a discrete valuation ring [4, Chapter 5, Theorem 8]. Hence $\text{gl dim } R = \text{gl dim } \hat{R} > 1$.

We conclude with an

EXAMPLE. Let $(R, \mathfrak{M})$ be a regular local ring of dimension 2, and let $\mathfrak{M} = (X_1, X_2)$. Assume that K , the quotient field of R , has characteristic $\neq 2$. Let Σ be the quaternion K -algebra $K[1, \alpha, \beta, \alpha\beta]$ defined by $\alpha^2 = X_1$, $\beta^2 = X_2$, and $\beta\alpha = -\alpha\beta$. Then Σ has no R -unramified splitting field.

PROOF. Let $\Lambda = R[1, \alpha, \beta, \alpha\beta]$. Λ is an R -order in Σ . $\text{Rad } \Lambda = \Lambda(\alpha, \beta)$, so that $\Lambda/\text{Rad } \Lambda \simeq R/\mathfrak{M}$. Since

$$0 \longrightarrow \Lambda\langle\beta, \alpha\rangle \longrightarrow \Lambda \oplus \Lambda \xrightarrow{(\alpha, \beta)} \Lambda(\alpha, \beta) \longrightarrow 0$$

is a minimal free Λ -resolution of $\text{Rad } \Lambda$, the projective dimension of $\text{Rad } \Lambda$ is 1, and hence $\text{gl dim } \Lambda = 2$.

Since $\text{Rad } \Lambda = \Lambda(\alpha, \beta) \not\supseteq \Lambda(X_1, X_2) = \mathfrak{M}\Lambda$, it follows that $\text{Rad } \Lambda \neq I\Lambda$ for any ideal I of R . Hence Λ is not R -separable [2, Corollary 3.2] and thus by our theorem, Σ has no R -unramified splitting field.

It should be noted that in the above example we restricted the global dimension of R to 2 for notational convenience only. If $\text{gl dim } R = n \geq 3$ and $\mathfrak{M} = (X_1, \dots, X_n)$ and if Λ is defined as above, then the only difference is that $\text{Rad } \Lambda = \Lambda(\alpha, \beta, X_3, \dots, X_n)$ and $\text{gl dim } \Lambda = n$.

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