THE SORGENFREY LINE IS NOT AN ELASTIC SPACE

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ABSTRACT. In this paper we show that the Sorgenfrey line is an example of a paracompact monotonically normal space which is not elastic. We also give a class of examples of elastic spaces which are not linearly stratifiable.

1. Introduction. H. Tamano and J. E. Vaughan conjecture in [3] that the closed continuous image of an elastic space is elastic. Since this closed continuous image is paracompact and monotonically normal (see [2], [3]), the question naturally arises as to whether a paracompact monotonically normal space is elastic. The purpose of this note is to answer this question in the negative by showing that the familiar *Sorgenfrey line* (the real line with the half-open interval topology) is a counterexample. We begin with the definition of an elastic space.

A pair-base $\mathcal{P} = \{P = (P_1, P_2)\}$ for a topological space X is a collection of pairs $P = (P_1, P_2)$ of subsets of X such that P_1 is open, and for each $x \in X$ and neighborhood U of x , there exists $(P_1, P_2) \in \mathcal{P}$ such that $x \in P_1 \subset P_2 \subset U$. Let $\mathcal{P}_1 = \{P_1 : P \in \mathcal{P}\}$, $\mathcal{P}_2 = \{P_2 : P \in \mathcal{P}\}$, and define $f: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ by $f(P_1) = P_2$. A T_1 -space is called an *elastic space* provided it has a pair-base $\mathcal{P}$ and a transitive relation $\mathcal{R}$ on $\mathcal{P}_1$ which is a framing of $\mathcal{P}_1$ (i.e., if $P_1, P'_1 \in \mathcal{P}_1$ and $P_1 \cap P'_1 \neq \emptyset$, then $P_1 \mathcal{R} P'_1$ or $P'_1 \mathcal{R} P_1$), and such that if $\mathcal{P}'_1 \subset \mathcal{P}_1$ has an upper bound (i.e., there is a $P_1 \in \mathcal{P}_1$ such that if $P'_1 \in \mathcal{P}'_1$ then $P'_1 \mathcal{R} P_1$) then $\text{Cl}(\bigcup \mathcal{P}'_1) \subset \bigcup f(\mathcal{P}'_1)$. A *linearly stratifiable space* may be defined as a T_1 -space with a linearly cushioned pair-base (i.e., the relation $\mathcal{R}$ is a linear order). Clearly every linearly stratifiable space is elastic.

2. Counterexample. *The Sorgenfrey line S is not an elastic space.* (In [2] it is shown that S is monotonically normal. S is well known to be paracompact.)

PROOF. Suppose S is elastic with elastic base $\mathcal{P}$ and framing $\mathcal{R}$ of $\mathcal{P}_1$. For each $x \in S$, choose some $(P_{1x}, P_{2x}) \in \mathcal{P}$ such that x is the left-hand endpoint of P_{ix} , $i = 1, 2$. Since S does not have a countable base, for some

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$\varepsilon > 0$ there are uncountably many P_{1x} such that $[x, x + \varepsilon) \subset P_{1x}$. Call this uncountable collection Q . Since S is separable, there is some $q \in S$ such that q is in uncountably many members of Q . Let $Q' = \{P_{1x} \in Q \mid q \in P_{1x}\}$. It follows that $\mathcal{R}$ restricted to Q' is a total relation (i.e., for each $(P_{1x}, P_{1y}) \in Q' \times Q'$, either $P_{1x} \mathcal{R} P_{1y}$ or $P_{1y} \mathcal{R} P_{1x}$). Therefore, it is easy to see that there is a cofinal subset $\mathcal{C}$ of Q' such that $\mathcal{R}$ restricted to $\mathcal{C}$ is a well-ordering. If $\mathcal{C}$ is countable, there is a $P_{1z} \in \mathcal{C}$ which bounds uncountably many elements of Q' . But then we can find a point $x \in S$ and points x_n converging to x from the right such that $P_{1x_n} \mathcal{R} P_{1z}$ (because S is hereditarily Lindelöf). Then $x \in \text{Cl}[\bigcup_{n=1}^{\infty} P_{1x_n}]$ and $x \notin \bigcup_{n=1}^{\infty} P_{2x_n}$, a contradiction.

Thus $\mathcal{C}$ must be uncountable. Let $\mathcal{C}' = \{P_{1x} \in \mathcal{C} \mid P_{1x} \text{ has at most countably many predecessors in } \mathcal{C}\}$. Then $\mathcal{C}'$ is uncountable and every countable subset of $\mathcal{C}'$ has an upper bound. There is a point $x \in S$, and points x_n converging to x from the right such that $P_{1x_n} \in \mathcal{C}'$. Thus $\{P_{1x_n}\}_{n=1}^{\infty}$ is bounded, $x \in \text{Cl}[\bigcup_{n=1}^{\infty} P_{1x_n}]$, but $x \notin \bigcup_{n=1}^{\infty} P_{2x_n}$, a contradiction. Therefore S is not elastic.

3. Elastic vs. linearly stratifiable. In [3], Tamano and Vaughan gave an example of an elastic space which is not stratifiable, but no example which is not linearly stratifiable. We can easily construct such examples, however, in the following way. Let X be the closed unit interval, and let Y be any linearly stratifiable space which is not stratifiable, e.g., the example in [3]. The disjoint union $X \vee Y$ of X and Y is obviously elastic. (Any topological sum (coproduct) of linearly stratifiable spaces is an elastic space.) There exist countable subsets of X , and hence of $X \vee Y$, which are not closed. Therefore by Proposition 2.10 of [4], if $X \vee Y$ is linearly stratifiable, it must be stratifiable. But then Y must be stratifiable, a contradiction. Thus $X \vee Y$ is not linearly stratifiable.

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