A CONDITION FOR THE EXISTENCE OF A STRONGLY EMBEDDED SUBGROUP

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ABSTRACT. Let D be a normal set of involutions in a finite group. Form the involutory graph with vertex set D by joining distinct commuting elements of D . Assume the product of any two such elements is in D , and the graph is disconnected. Then the group generated by D contains a strongly embedded subgroup. Two corollaries are proved.

Let G be a finite group and D a collection of involutions normalized by G . Form the graph $\mathcal{D} = \mathcal{D}(D)$ with vertex set D and edges (u, v) where $uv = vu \neq 1$.

THEOREM 1. Assume $\mathcal{D}$ is disconnected and for any edge (u, v) in $\mathcal{D}$, $uv \in D$. Let H be the normalizer in G of a connected component of $\mathcal{D}$. Then $H \cap \langle D \rangle$ is strongly embedded in $\langle D \rangle$.

The following two theorems are corollaries to Theorem 1 and Bender's classification of groups with a strongly embedded subgroup.

THEOREM 2. Let G be a transitive permutation group on a set Ω , let $\alpha \in \Omega$, and let $H = G_\alpha$. Assume Q is a normal subgroup of even order in H such that every involution z in Q fixes exactly one point of Ω . Set $L = \langle z^G \rangle$. Then either

- (1) $L \leq Z^*(G)$ and $O(L)$ is transitive on Ω , or
- (2) L^Ω is $L_2(q)$, $Sz(q)$, or $U_3(q)$, $q > 2$ even, in its natural doubly transitive representation.

THEOREM 3. Let z^G be a conjugacy class of involutions in the group G , set $W = \langle z^G \cap C(z) \rangle$ and $L = \langle z^G \rangle$. Assume W is abelian and $W^\#$ is fused in G and $W \not\leq G$. Then either

- (1) $L \leq Z^*(G)$, or
- (2) L is isomorphic to $L_2(q)$, $Sz(q)$, or $U_3(q)$, $q > 2$ even, or the perfect central extension of Z_3 by $U_3(q)$, $q \equiv -1 \pmod{3}$ even.

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Theorem 3 can be used to significantly shorten the proof of an important result of Shult [5], classifying finite groups in which the weak closure of a central involution in its centralizer is abelian. Theorem 2 contains theorems of Shult ([5], [6]) and Hering [4] on transitive permutation groups.

The proof of Theorem 1 depends on the following four results:

THEOREM 4 (BENDER [2]). *Let H be a strongly embedded subgroup of the finite group G . Then either G has 2-rank one or $O^2(G)/O(G)$ is isomorphic to $L_2(q)$, $Sz(q)$, or $U_3(q)$, $q > 2$ even.*

THEOREM 5 (STEINBERG, ALPERIN AND GORENSTEIN, GRIESS [3]). *Let $G \cong L_2(q)$, $Sz(q)$, or $U_3(q)$, $q > 2$ even, and let G^* be the covering group of G . Then $G = G^*$ except in the following cases:*

- (1) $G \cong L_2(4)$ and $G^* \cong SL_2(5)$,
- (2) $G \cong Sz(8)$ and $Z(G^*) \cong Z_2 \times Z_2$,
- (3) $G \cong U_3(q)$, $q \equiv -1 \pmod{3}$, and $Z(G^*) \cong Z_3$.

THEOREM 6 [1]. *Let G be a finite group, H a proper subgroup of G and z an involution in the center of a Sylow 2-subgroup of G . Assume*

- (1) $z \in H^g$ if and only if $g \in H$, and
- (2) if u is an involution with $z \in C(u) \not\leq H$ then $H \cap L$ is strongly embedded in $L = \langle z^G \cap C(u) \rangle$.

Then $H \cap \langle z^G \rangle$ is strongly embedded in $\langle z^G \rangle$.

LEMMA 7. *Assume the hypothesis of Theorem 1. Let Δ be the connected component of $\mathcal{D}$ normalized by H , let $z \in \Delta$, and let S be a Sylow 2-subgroup of $C(z)$. Then*

- (1) G^D is transitive,
- (2) $z \in H^g$ if and only if $g \in H$,
- (3) S is Sylow in G .

PROOF. Let $u \in D$ and assume uz has even order. Let v be the involution in $\langle uz \rangle$. Thus $vz \in D$, so by hypothesis $v = (vz)z \in D$. So u , v , and z are all in Δ .

Let $w \in D - \Delta$ and $x \in \Delta$. Then by the last paragraph, wx and wz have odd order, so w is conjugate to x in $\langle w, x \rangle$ and to z in $\langle w, z \rangle$. Therefore G^D is transitive.

Assume S is not Sylow in G . Then S is of index 2 in some 2-subgroup T of G . Let $t \in T - S$. Then $z \neq z^t \in Z(S)$, so $u = zz^t \in D \cap Z(T)$, impossible as G^D is transitive. Notice that $z^S = z \in \Delta$, so $S \leq H$.

Finally let $u \in H \cap D$. As S is Sylow in H we may assume $u \in S$. Thus $[u, z] = 1$, so $u \in \Delta$. Then $H \cap D = \Delta$, implying (2).

Now for the proof of Theorem 1. Assume the theorem to be false and let G be a minimal counterexample. First assume x is an involution in

the center of G , and set $\bar{G} = G/\langle x \rangle$. Let $(\bar{u}, \bar{v})$ be an edge in $\mathcal{D}(\bar{D})$. Then either (u, v) is an edge in $\mathcal{D}$ and thus $uv \in D$, or $(uv)^2 = x$. In the latter case $ux \in D$, so $x \in D$, impossible as G^D is transitive. So $uv \in D$ and thus $\bar{u}\bar{v} \in \bar{D}$. It follows that $\bar{G}$ satisfies the hypothesis of Theorem 1, so by minimality of G , $\bar{H} \cap \bar{L}$ is strongly embedded in $\bar{L} = \langle \bar{D} \rangle$. By Theorem 4, $L/O(L) \cong Z_2$, $L_2(q)$, $Sz(q)$, or $U_3(q)$, $q > 2$ even. It follows from Theorem 5 that either $L = \langle x \rangle \times K$ with $K \cong \bar{L}$, or $L/O(L) \cong SL_2(5)$ or $(Sz)^\wedge(8)$, a perfect extension of $\langle x \rangle$ by $Sz(8)$. $L/O(L) \cong SL_2(5)$ as L contains an involution $z \neq x$. If $L/O(L) \cong (Sz)^\wedge(8)$ then $x = zu$ for some $u \in D$, so $x \in D$, a contradiction. So $L = \langle x \rangle \times K$. Further if u and v are distinct commuting involutions in $L - (K \cup \{x\})$, then $uv \notin D$, so $K = \langle D \rangle$. But $H \cap K$ is strongly embedded in K , a contradiction.

It follows that $Z(G)$ has odd order. Next suppose u is an involution centralizing $z \in H \cap D$ with $C(u) \not\leq H$. Then $u \notin Z(G)$ so $C(u) \neq G$. But $C(u)$ satisfies the hypothesis of Theorem 1, so minimality of G implies $H \cap L$ is strongly embedded in $L = \langle C(u) \cap D \rangle$. But now Theorem 6 contradicts the choice of G .

Finally the proofs of Theorems 2 and 3. In Theorem 2, H is the stabilizer of a point of Ω ; in Theorem 3 set $H = N(W)$. In Theorem 2 let z be an involution in $\mathcal{Q}$. Set $X = O(G)$ and $L = \langle z^G \rangle$. By Theorem 1, $H \cap L$ is strongly embedded in L . We may assume $z \notin Z^*(G)$. Thus there exists a 4-group U containing z with $U^\# \subseteq z^G$. Then $X = \bigcap_{U^\#} C_X(u) \leq H$. Therefore in Theorem 2, X fixes Ω pointwise, so $X = 1$. And in Theorem 3, $[X, W] \leq X \cap W = 1$, so $X \cap L \leq Z(L)$. Thus Theorems 4 and 5 yield the desired conclusion.

REFERENCES

1. M. Aschbacher, *Finite groups with a proper 2-generated core* (to appear).
2. H. Bender, *Transitive Gruppen gerader Ordnung in denen jede Involution genau eine Punkt festlegt*, J. Algebra **17** (1971), 527–554.
3. W. Feit, *The current situation in the theory of finite simple groups*, Proc. Internat. Congress Math. (Nice, 1970), vol. 1, Gauthier-Villars, Paris, 1971, pp. 55–93.
4. C. Hering, *On subgroups with trivial normalizer intersection*, J. Algebra **20** (1972), 622–629.
5. E. Shult, *On the fusion of an involution in its centralizer* (to appear).
6. ———, *On a class of doubly transitive groups*, Illinois J. Math. **16** (1972), 434–445.

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