KUREPA'S HYPOTHESIS AND σ -COMPLETE IDEALS¹

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ABSTRACT. A set-theoretic combinatorial hypothesis, known as Kurepa's hypothesis, is used to obtain results concerning the saturatedness of ideals and families of measures.

DEFINITION. $\text{Sat}(\mathcal{T})$ holds if

- (1) for some set S , $\mathcal{T}$ is an ideal in $\mathcal{P}(S)$;
- (2) $\mathcal{T}$ is closed under countable unions;
- (3) all finite sets belong to $\mathcal{T}$, $S \notin \mathcal{T}$;
- (4) for every family $\mathcal{F} \subseteq \mathcal{P}(S) - \mathcal{T}$, if for all $X, Y \in \mathcal{F}$; either $X = Y$ or $X \cap Y \in \mathcal{T}$, then $|\mathcal{F}| \leq \aleph_1$.

Solovay showed by a complicated argument, which has never been published, the following

THEOREM 1. *If the axiom of constructibility $V=L$ holds, then there is no $\mathcal{T}$ such that $\text{Sat}(\mathcal{T})$.*

The main part of the proof consists in showing that Theorem 1 holds for $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$. We shall show that this follows from so-called Kurepa's hypothesis (KH). A more recent result of Solovay states that $V=L$ implies KH. Hence Theorem 1 for $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$ will be established. A proof of a version of KH from $V=L$ can be found in [3]. Although in some respect our proof of Theorem 1 gives less than Solovay's original proof,² or subsequent extensions of Kunen, we shall obtain a little bit more in a different direction (see corollary to Lemma 2). This will enable us to use KH to obtain a result concerning the following problem of Ulam: Can one define $\aleph_1$ σ -additive measures on $\mathcal{P}(\omega_1)$ such that each set is measurable with respect to at least one of them? It is also required that all finite sets have measure zero and ω_1 has measure one with respect to all measures.

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² Actually, the main result of Solovay is stronger than Theorem 1. He has shown that if there is a $\mathcal{T}$ such that $\text{Sat}(\mathcal{T})$, then so-called $0^\#$ exists. A proof of this and of other related results appears in Kunen [8] (see Theorems 11.12, 11.13).

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We shall adopt the following version of KH: There exist functions $f_\alpha \in \omega^{\omega_1}$, $\alpha \in \omega_2$, such that for all $\alpha, \beta \in \omega_2$, if $\alpha \neq \beta$, then

$$|\{\eta: f_\alpha(\eta) = f_\beta(\eta)\}| \leq \aleph_0.$$

This is a very weak version of KH and can easily be deduced from all other variants of KH.³

The following lemma is analogous to a lemma of Ulam [6].

LEMMA 1. *KH implies the existence of a family of sets A_α^n , $n \in \omega$, $\alpha \in \omega_2$ such that*

- (i) *for every $\alpha \in \omega_2$, $\bigcup \{A_\alpha^n: n \in \omega\} = \omega_1$,*
- (ii) *for every $n \in \omega$ and every $\alpha, \beta \in \omega_2$, if $\alpha \neq \beta$, then $|A_\alpha^n \cap A_\beta^n| \leq \aleph_0$.*

PROOF. Define $\eta \in A_\alpha^n$ iff $f_\alpha(\eta) = n$. Both (i), (ii) are obvious.

LEMMA 2. *Assume KH. Then there is no $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$ such that $\text{Sat}(\mathcal{T})$.*

PROOF. Let $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$ satisfy (1), (2), (3) of Definition 1. Let A_α^n be as in Lemma 1. For each α there is some n_α such that $A_\alpha^{n_\alpha} \notin \mathcal{T}$. If not, then by (2) and Lemma 1(i), $\omega_1 \in \mathcal{T}$. This contradicts (3) ($\omega_1 = S$).

Hence there is $\bar{n} \in \omega$ and $B \subseteq \omega_2$ such that $|B| = \aleph_2$ and $n_\alpha = \bar{n}$ for all $\alpha \in B$. Let $\mathcal{F} = \{A_\alpha^{\bar{n}}: \alpha \in B\}$. Then $|\mathcal{F}| = \aleph_2$, $\mathcal{F} \subseteq \mathcal{P}(\omega_1) - \mathcal{T}$, and for all $X, Y \in \mathcal{F}$, either $X = Y$, or $|X \cap Y| \leq \aleph_0$. Hence (4) fails for $\mathcal{T}$ and Lemma 2 is proved.

We have the following corollary of the proof of Lemma 2.

COROLLARY. *Assume KH. Let $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$ and satisfy (1), (2), (3). Then there is a family $\mathcal{F} \subseteq \mathcal{P}(\omega_1) - \mathcal{T}$ such that $|\mathcal{F}| = \aleph_2$ and for all $X, Y \in \mathcal{F}$, either $X = Y$ or $|X \cap Y| \leq \aleph_0$.*

The conclusion $|X \cap Y| \leq \aleph_0$ is stronger than $X \cap Y \in \mathcal{T}$.

The rest of the proof, for $|S| > \aleph_1$, is handled by familiar arguments due to Tarski and Ulam. It can be deduced from Lemma 2 that if $\text{Sat}(\mathcal{T})$ holds for some $\mathcal{T} \subseteq \mathcal{P}(S)$, then $\mathcal{T}$ is closed under $\aleph_1$ -unions. But if λ is the least cardinal such that there is a $\mathcal{T} \subseteq \mathcal{P}(\lambda)$ closed under $\aleph_1$ -unions and satisfying $\text{Sat}(\mathcal{T})$, then either λ is weakly inaccessible and $2^{\omega_1} \geq \lambda$, or λ is a measurable cardinal. (See e.g. [2].) Since we are assuming $V = L$, the first case is impossible: $V = L$ implies GCH. Also, by a theorem of Scott [5], $V = L$ implies that there is no measurable cardinal. Hence Theorem 1 follows.

We shall now turn our attention to Ulam's problem.

³ A recent unpublished result of Baumgartner is that this is in fact properly weaker than the well-known form of Kurepa's hypothesis.

THEOREM 2. Assume KH. Let μ_α , $\alpha \in \omega_1$, be a family of σ -additive measures defined on σ -fields $\mathcal{S}_\alpha \subseteq \mathcal{P}(\omega_1)$ such that

(a) $\mu_\alpha(\{x\})=0$ for every $x \in \omega_1$ and every $\alpha \in \omega_1$;

(b) $\mu_\alpha(\omega_1)=1$ for every $\alpha \in \omega_1$.

Then there is an $X \subseteq \omega_1$ such that $X \notin \mathcal{S}_\alpha$ for uncountably many α 's, i.e. X is nonmeasurable with respect to uncountably many μ_α 's.

PROOF. We assume the contrary. Hence every $X \subseteq \omega_1$ is measurable with respect to all but countably many μ_α 's. We shall define an ideal $\mathcal{T} \subseteq \mathcal{P}(\omega_1)$ as follows: $X \in \mathcal{T}$ iff

$$(*) \quad |\{\alpha: \mu_\alpha(X) > 0 \text{ or } X \notin \mathcal{S}_\alpha\}| \leq \aleph_0.$$

We claim that $\mathcal{T}$ satisfies (1), (2), (3) of the definition of $\text{Sat}(\mathcal{T})$, with $S = \omega_1$. We check only (2). Let $X_n \in \mathcal{T}$ for all $n \in \omega$. Hence by (*) there are α_n , $n \in \omega$, such that $\mu_{\eta}(X_n)=0$ for $\eta \geq \alpha_n$. Let $\alpha = \sup\{\alpha_n: n \in \omega\}$. Since ω_1 is regular, $\alpha \in \omega_1$. If $\eta \geq \alpha$, then $\mu_\eta(X_n)=0$ for all $n \in \omega$. Hence

$$\mu_\eta(\bigcup \{X_n: n \in \omega\}) = 0 \quad \text{for all } \eta \geq \alpha,$$

i.e. $\bigcup \{X_n: n \in \omega\} \in \mathcal{T}$.

Let $\mathcal{F} \subseteq \mathcal{P}(\omega_1) - \mathcal{T}$ be as in the corollary to Lemma 2. By easy cardinality considerations we can find $\mathcal{G} \subseteq \mathcal{F}$ and $\bar{\alpha} \in \omega_1$ such that $|\mathcal{G}| = \aleph_2$ and for all $X \in \mathcal{G}$, $X \in \mathcal{S}_\eta$ for all $\eta \geq \bar{\alpha}$.

Let $X \in \mathcal{G}$. Since $X \notin \mathcal{T}$, there is a $\beta \geq \bar{\alpha}$ such that $\mu_\beta(X) > 0$. Hence we find $\mathcal{H} \subseteq \mathcal{G}$ and $\bar{\beta} \geq \bar{\alpha}$ such that $|\mathcal{H}| = \aleph_2$ and $\mu_{\bar{\beta}}(X) > 0$ for all $X \in \mathcal{H}$.

Since any two distinct members of $\mathcal{F}$ have at most countable intersection, we have $\mu_{\bar{\beta}}(X \cap Y) = 0$ for any two distinct $X, Y \in \mathcal{H}$.

The existence of an $\mathcal{H}$ as described above is clearly contradictory. Hence Theorem 2 is proved.

We do not know whether KH implies the negative answer to Ulam's question. By a Theorem of Alaoglu and Erdős [1], there does not exist a family of countably many measures on $\mathcal{P}(\omega_1)$ such that every set is measurable with respect to at least one of them. For another approach to Ulam's problem see [4], where it is shown to be consistent with the axioms of $\text{ZF} + \text{AC}$, that $\aleph_1$ 0, 1-valued measures are not sufficient. For the sake of completeness, we also observe the following. Theorem 2 will remain true if we replace the assumption of KH by CH (the continuum hypothesis). This follows from the results of S. Banach and C. Kuratowski [7]. They prove the following theorem. If $2^{\aleph_0} = \aleph_1$, then there exist sets A_n , $n \in \omega$, such that for every measure μ as in Theorem 2, at least one of A_n is nonmeasurable with respect to μ . Hence given $\aleph_1$ measures μ_α , $\alpha \in \omega_1$, one A_n is nonmeasurable with respect to $\aleph_1$ of them.

It should also be remarked that Theorem 2 will be true without any hypothesis, if it is required that μ_α 's take values 0, 1 only. This follows from the fact that there exists a family of sets $A_n \subseteq \omega_1$, $n \in \omega$, such that for any 0, 1-valued measure μ as in Theorem 2, not all A_n can be measurable with respect to μ .

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