A NOTE ON PATHS THROUGH O

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ABSTRACT. We show that a hyperarithmetical set can be truth table reduced to a Π_1^1 -path through O iff it is truth table reducible to some r.e. set.

It is known from results of Feferman and Spector [1] that while there are no Σ_1^1 -paths through O , there do exist Π_1^1 such paths. Here by a path is meant a linearly ordered subset P of O , closed under $<_O$ and having order type ω_1 .

In this note we prove

THEOREM 1. *If P is a Π_1^1 -path through O , A is hyperarithmetical, and A is truth table reducible to P , then the Turing degree of A is at most $0'$.*

Thus in a certain sense such paths contain very little "information". It is not known if a hyperarithmetical set of Turing degree greater than $0'$ can be Turing reduced to such a path.

PROOF. We shall actually prove a somewhat stronger fact. If P is a Π_1^1 -path through O and A is truth table reducible to P , then either A is truth table reducible to a proper segment of P (which is necessarily r.e.) or else *all* of P is used in an essential way, and then one can go backwards and arithmetically decide P from A .

So suppose the Turing degree of A is not $\leq 0'$ and A is truth table reducible to such a path P . Then there exists a z such that, for all n , $C_A(n) = U(\mu y T^P(z, n, y))$ and moreover, for all $X \subseteq N$, $U(\mu y T^X(z, n, y))$ is a total function of n . (See [4, p. 143, Theorem XIX].)

Now by (1) there exists a $d \in O^*$ and an r.e. ordering $\leq$ such that $P \subseteq P_1 = \{y \mid y \leq d\}$, $\leq$ is a linear ordering on P_1 and P is the maximal well ordered segment of P_1 . (In fact, $\leq$ is that r.e. linear ordering which restricts on O to $<_O$.)

Consider pairs (a, b) , $a, b \in P_1$, $a < b$, and consider a z -computation of $C_A(n)$ where: whenever the machine asks, "does $m \in P$?" the answer given is "yes" if $m \leq a$, "no" if $m \notin P_1$ or $b \leq m$, and no answer is given if neither condition holds. We will say that (a, b) is *adequate* for n if the correct value of $C_A(n)$ is computed in this way.

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Clearly, there is an (a, b) adequate for a given n and in fact for finitely many n . For consider the *actual* computations of $C_A(n_i)$, $i=1, \dots, k$, from P itself. Let $a = \max$ relative to $\leq$ of all yes answers given and $b = \min$ of all no answers, then (a, b) is adequate for $n_1, \dots, n_k$. In this case we shall have $a \in P$, $b \notin P$. However, it is conceivable that (a, b) is adequate for the n_i 's and yet both (or neither) of a, b are in P . Note that if (a, b) is adequate for n , $a \leq a' < b' \leq b$ then (a', b') is adequate for n .

We will say that (a, b) is *good*, if for all $F \subseteq N$, F finite, there are a', b' , $a \leq a' < b' \leq b$ such that (a', b') is adequate for (every element of) F . The predicate $G(a, b)$ ((a, b) is good) is arithmetical in A and hence hyperarithmetical.

Now we claim that if $G(a, b)$, then $\exists X, \{y | y \leq a\} \subseteq X \subseteq \{y | y \leq b\}$ and $C_A(n) \equiv U(\mu y T^X(z, n, y))$.

For note that if $a < c < b$ and $G(a, b)$ then either $G(a, c)$ or $G(c, b)$. Otherwise there are finite sets F_1, F_2 such that no pair (a', b') contained in (a, c) is adequate for F_1 , and no pair (a', b') contained in (c, b) is adequate for F_2 . But then no pair (a', b') contained in (a, b) is adequate for $F_1 \cup F_2$ contradicting the goodness of (a, b) . (Either (a', b') is contained in (a, c) or in (c, b) or else (a', c) is adequate for F_1 .)

Now given $G(a, b)$, let $c_1, c_2, \dots$ be an enumeration of P_1 . Define (a_i, b_i) by:

$$\begin{aligned} (a_0, b_0) &= (a, b), \\ (a_{n+1}, b_{n+1}) &= (a_n, b_n) \quad \text{unless } a_n < c_{n+1} < b_n, \\ &= (a_n, c_{n+1}) \quad \text{if } a_n < c_{n+1} < b_n \text{ and } G(a_n, c_{n+1}), \\ &= (c_{n+1}, b_n) \quad \text{otherwise.} \end{aligned}$$

Then $G(a_n, b_n)$ holds for all n and no x can satisfy $a_n < x < b_n$ for all n . Let X be the set $\{x | (\exists n)(x \leq a_n)\}$. Claim $U(\mu y T^X(z, n, y))$ always equals $C_A(n)$.

Otherwise (by choice of z) there is a p such that $U(\mu y T^X(z, p, y))$ is defined and unequal to $C_A(p)$. Consider a , the maximum (in $\leq$) of all the "yes" answers during this computation from X . Similarly b is the minimum of all "no" answers for elements of P_1 . Then there is an n such that $a \leq a_n < b_n \leq b$. But clearly (a_n, b_n) cannot be good since no pair (a', b') contained in (a, b) can be adequate for $\{p\}$.

Thus $G(a, b)$ implies $b \notin P$. Otherwise an X such as above would be a proper segment of P , and hence r.e. But clearly $b \in P_1 - P$ implies that for all $a \in P$, $G(a, b)$. Hence we get

$$x \in P \leftrightarrow x \in P_1 \wedge \neg (\exists a)(G(a, x)),$$

a contradiction since P is not hyperarithmetical. Q.E.D.

It is fairly straightforward to show that the lower bound on the degree of A is best possible.

THEOREM 2.¹ *There is a Π_1^1 -path P through O such that the set $A = \{n \mid (\exists y)T(n, n, y)\}$ is 1-1 reducible to P .*

PROOF. Define an ordering R on N by

$$R(x, z) \leftrightarrow (\exists y)(T(x, x, y) \wedge (\forall u \leq y)(\neg T(z, z, u))).$$

This is a partial ordering, the elements in A are a sequence of type ω and the elements not in A are a set of incomparables above this sequence. Now apply the construction of [3, p. 45] to this ordering. We get a function g from N to O such that $g[N-A]$ is a set of incomparables and if $x \in A$, $y \in N-A$ then $g(x) <_O g(y)$. Take any $x_0 \in N-A$ and pick $d \in O^*$ such that $g(x_0) <_O d$. The path P corresponding to d has the required property.

REMARK. Note that A is in fact 1-1 reduced to a proper segment S of P . $P-S$ plays no role, since $g[A] \subseteq S$ and $g[N-A] \subseteq N-P$.

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¹ This result was also found, independently and simultaneously by C. Jockusch.