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THE NUMBER OF FIELD TOPOLOGIES ON COUNTABLE FIELDS

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ABSTRACT. J. O. Kiltinen proves that every infinite field admits a nondiscrete, Hausdorff field topology. In this note it is shown that every countable field K admits $2^{2^{\aleph_0}}$ many field topologies, which even fail to be the join of locally bounded ring topologies.

1. **Introduction.** In §2 we give a method for generating a fundamental system $\{V_n \mid n \in \omega\}$ of neighborhoods of zero for a field topology on K. To do this, we first define the notion of a condition. This is a function from $\omega \times \{0, 1\}$ into the set of finite subsets of K with some further properties. A condition p decides for a finite number of elements of K if they are elements of V_n or not by saying r is an element of V_n if $r \in p(n, 0)$ and r is not an element of V_n if $r \in p(n, 1)$. Given two conditions p and p', then p' *extends* p if $p(n, i) \subset p'(n, i)$ for every (n, i). If G is a chain of conditions, then by the above decision process we get a fundamental system of a field topology.

Since a condition decides only for a finite number of elements of K if they are elements of V_n or not, we can prove in §3 that there are "many" possibilities to extend a condition. In §4 we define what it means for a set of chains of conditions to be entwined. If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are different nonempty subsets of an entwined set then the join-topologies $\bigvee \{\mathscr{T}_G \mid G \in \mathfrak{G}_1\}$ and $\bigvee \{\mathscr{T}_G \mid G \in \mathfrak{G}_2\}$ are also different. Using the results of §3 we can construct an entwined set of power $2^{\aleph_0}$ of chains of conditions in such a way that for every subset $\mathfrak{G}_1$ the topology $\bigvee \{\mathscr{T}_G \mid G \in \mathfrak{G}_1\}$ is not the join of locally bounded ring topologies.

2. **Chains of conditions.** Let $(K, +, \cdot, 0, 1)$ be a countable field and let ϕ_K denote the set of functions φ_d, φ_c, φ_b and φ_a, $a \in K$, defined by $\varphi_d(X) = X/(1-X)$, $\varphi_c(X) = X \cdot X$, $\varphi_b(X) = X - X$ and $\varphi_a(X) = a \cdot X$, for every subset X of K with $1 \notin X$. Since K is countable there is a sequence $(\varphi_n)_{n \in \omega}$ of elements of ϕ_K such that the set $\{n \mid \varphi = \varphi_n\}$ is infinite for every $\varphi \in \phi_K$. A sequence $\{V_n \mid n \in \omega\}$ of subsets of K is called a fundamental system, if $1 \notin V_n$, $V_{n+1} \subset V_n$, $\varphi_n(V_{n+1}) \subset V_n$. For every field topology there is a basic

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system of neighborhoods of zero which is a fundamental system, and every fundamental system determines a field topology.

1. DEFINITION. A function p from $\omega\times\{0, 1\}$ into the set of all finite subsets of K is called a condition, if the following properties hold:

(a) $0 \in p(n, 0)$ and $1 \in p(n, 1)$,

(b) $p(n, 0)\cap p(n, 1)=\varnothing$,

(c) $p(n+1, i)\subset p(n, i)$,

(d) $\varphi_n(p(n+1, 0))\subset p(n, 0)$.

Let P be the set of all conditions. P is not empty, since p^0 defined by $p(n, i)=\{i\}$ for every $n \in \omega$, is an element of P. If p and p' are two conditions we say that p' *extends* p (written $p\leqq p'$), if $p(n, i)$ is a subset of $p'(n, i)$ for every $(n, i) \in \omega\times\{0, 1\}$. $\leqq$ is a partial ordering of P. If G is a chain of conditions, then V_n^G is defined to be $\bigcup \{p(n, 0)|p \in G\}$.

2. THEOREM. *Let G be a chain of conditions, then* $\{V_n^G|n \in \omega\}$ *is a fundamental system.*

The proof is straightforward if we use the fact that

$$\varphi_n(\bigcup \{p(n + 1, 0) \mid p \in G\}) = \bigcup \{\varphi_n(p(n + 1, 0)) \mid p \in G\}.$$

For every chain of conditions let $\mathscr{T}_G$ denote the field topology which is determined by $\{V_n^G|n \in \omega\}$.

3. THEOREM. *If* $\mathscr{T}$ *is a field topology with a countable basis, then there is a chain G of conditions such that* $\mathscr{T}=\mathscr{T}_G$.

PROOF. Let $\{V_n|n \in \omega\}$ be a fundamental system which determines the topology $\mathscr{T}$ and let $(r_k^n)_{k\in\omega}$ be a well ordering of V_n for each $n \in \omega$. By recursion we define for every m a condition p_m as follows:

$$\begin{aligned} p_m(n, i) &= p_m(n + 1, i) \cup \varphi_n(p_m(n + 1, i)) \cup \{r_j^n \mid 0 \leqq j \leqq m - n\} \\ &\qquad\qquad\qquad \text{if } n \leqq m \quad \text{and} \quad i = 0, \\ &= \{i\} \quad \text{otherwise.} \end{aligned}$$

Let G be the set $\{p_m|m \in \omega\}$. Then we have for every n that $V_n^G=V_n$ and therefore is $\mathscr{T}_G=\mathscr{T}$. □

3. **Extensions of conditions.** Here we prove that for every condition p, for each $(n, i) \in \omega\times\{0, 1\}$ and for nearly all (this means that for all but finitely many) $r \in K$ there exists a condition $p_r\geqq p$ such that $r \in p_r(n, i)$. This will enable us to prove in §3 that there are "many" different chains of conditions. First the easy case.

4. THEOREM. *Let p be a condition and* $n \in \omega$. *Then for nearly all* $r \in K$ *there is a condition* p_r *such that* $p_r\geqq p$ *and* $r \in p_r(n, 1)$.

PROOF. Let $r \in K$, $r \notin p(0, 0)$. If we define p_r as follows:

$$\begin{aligned} p_r(n, i) &= p(n, i) \cup \{r\} && \text{if } i = 1, \\ &= p(n, i) && \text{otherwise,} \end{aligned}$$

then p_r has the desired properties. Since $p(0, 0)$ is finite, this holds for nearly all r. □

Now the other case. Let $R(K)$ be the set of all rational functions over K. If H is a subset of $R(K)$ and if for each $f \in H$, $r \in K$ is in the domain of f, then $H(r)$ shall denote the set of all $f(r)$, with $f \in H$. Let $f \in R(K)$ and $a \in K$ such that $f(0) \neq a$. Then for nearly all $r \in K$, $f(r) \neq a$. So we obtain:

5. LEMMA. *Let $H \subset R(K)$ and $M \subset K$ be finite. If $H(0) \cap M = \varnothing$, then for nearly all $r \in K$, $H(r) \cap M = \varnothing$.*

6. LEMMA. *Let $\varphi \in \phi_K$ and let H be a finite subset of $R(K)$ with $1 \notin H(0)$, then there is a finite $H' \subset R(K)$ such that, for nearly all $r \in K$, $\varphi(H(r)) = H'(r)$.*

PROOF. If $\varphi = \varphi_d$ we define H' to be $\{f/(1-g) \mid f, g \in H\}$. Since $1 \notin H(0)$ we have for nearly all $r \in K$, $1 \notin H(r)$. Let r be such an element of K. Then $\varphi_d(H(r)) = H(r)/(1-H(r)) = H'(r)$. Thus, for nearly all $r \in K$, $\varphi_d(H(r)) = H'(r)$. The proof is similar if $\varphi = \varphi_c$, φ_b or φ_a. □

Now let $p \in P$ and let H be a finite subset of $R(K)$. By induction over n we can prove

7. THEOREM. *If $H(0)$ is a subset of $p(n, 0)$, then for nearly all $r \in K$ there is a condition $p_r \geq p$ such that:*

(1) $H(r) \subset p_r(n, 0)$,

(2) $p_r(m, i) = p(m, i)$ *if* $m > n$ *or* $i = 1$.

(i) The Theorem holds for $n = 0$.

PROOF. Since $H(0)$ is a subset of $p(n, 0)$, $H(0) \cap p(0, 1) = \varnothing$. By Lemma 5, we have for nearly all $r \in K$ that $p(0, 1) \cap H(r) = \varnothing$. Let r be such an element. If p_r is defined by

$$\begin{aligned} p_r(m, i) &= p(m, i) \cup H(r) && \text{if } m = 0 \text{ and } i = 0, \\ &= p(m, i) && \text{otherwise,} \end{aligned}$$

then p_r has the desired properties.

(ii) Assume the Theorem holds for n, then it holds for $n+1$.

PROOF. First we choose a finite subset H''' of $R(K)$, with $H'''(0) \subset p(n, 0)$ as follows: Let $H' = H \cup \{f_a \mid a \in p(n+1, 0)\}$, where f_a is the function defined by $f_a(r) = a$ for every $r \in K$. Since $1 \notin p(n+1, 0)$ and $H'(0) \subset p(n+1, 0)$, we have by Lemma 6 that there is a finite $H'' \subset R(K)$ such that for nearly all $r \in K$, $H''(r) = \varphi_n(H'(r))$. Let L be the set of these r's. If we define $H''' = H' \cup H''$, then we have that $H'''(0) \subset p(n, 0)$. By assumption there are for nearly all $r \in K$ conditions $p'_r \geqq p$ such that:

(1) $H'''(r) \subset p'_r(n, 0)$,

(2) $p(m, i) = p'_r(m, i)$ if $m > n$ or $i = 1$.

Let L' be the set of these r's and let $r \in L \cap L'$. Then we define p_r by

$$p_r(m, i) = p'_r(m, i) \cup H(r) \qquad \text{if } m = n+1 \text{ and } i = 0,$$
$$\qquad\quad = p'_r(m, i) \qquad\qquad \text{otherwise.}$$

p_r has the desired properties. Since nearly all $r \in K$ are in $L \cap L'$, the theorem holds for $n+1$.

8. Corollary. *Let p be a condition and $n \in \omega$. Then for nearly all $r \in K$ there are conditions $p_r \geqq p$, such that $r \in p_r(n, 0)$.*

Proof. Take H to be {id}, where id is the function which maps every element of K onto itself. By Theorem 7 we get the desired result. □

4. **Entwined sets of chains of conditions.** To prove that there are $2^{2^{\aleph_0}}$ many field topologies on K, it suffices to show that there is a set $\mathfrak{G}$ of power $2^{\aleph_0}$ of chains of conditions such that for any two different non-empty subsets $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of $\mathfrak{C}$ the join-topologies $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_1\}$ and $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_2\}$ are also different.

9. Definition. Let $\mathfrak{G}$ be a set of chains of conditions. $\mathfrak{G}$ is called *entwined* if, for every $n \in \omega$ and for every finite subset $\{G_i | 0 \leqq i \leqq m\}$ of $\mathfrak{G}$, there are conditions $p_i \in G_i$, $0 \leqq i \leqq m$, such that

$$\bigcap \{p_i(n, 0) \mid 1 \leqq i \leqq m\} \cap p_0(0, 1) \qquad \text{is not empty.}$$

An easy consequence of Definition 9 is that there is a sequence which converges to zero in all of the topologies $\mathscr{T}_{G_1}, \cdots, \mathscr{T}_{G_m}$ but which is bounded away from zero in $\mathscr{T}_{G_0}$.

10. Theorem. *If $\mathfrak{G}$ is entwined, then $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G} \backslash \{G_0\}\}$ is not finer than $\mathscr{T}_{G_0}$ for every $G_0 \in \mathfrak{G}$.*

Proof. Suppose $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G} \backslash \{G_0\}\}$ is finer than $\mathscr{T}_{G_0}$ for some $G_0 \in \mathfrak{G}$. Then there are $G_1, \cdots, G_m \in \mathfrak{G}$ and $k_1, \ldots, k_m \in \omega$ such that

$$\bigcap \{V_{k_i}^{G_i} \mid 1 \leqq i \leqq m\} \subset V_0^{G_0}.$$

Let $k_0 = \max\{k_i | 1 \leqq i \leqq m\}$. Since $\mathfrak{G}$ is entwined, there are conditions $p_i \in G_i$, $0 \leqq i \leqq m$, such that $M = \bigcap \{p_i(k_0, 0) | 1 \leqq i \leqq m\} \cap p_0(0, 1)$ is not empty. If $r \in M$, then $r \in V_{k_0}^{G_i} \subset V_{k_i}^{G_i}$ and $r \in V_0^{G_0}$. This is a contradiction. □

11. COROLLARY. *Let $\mathfrak{G}$ be entwined and $\mathfrak{G}_1, \mathfrak{G}_2$ nonempty different subsets of $\mathfrak{G}$. Then $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_1\}$ and $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_2\}$ are different.*

PROOF. Let $\mathfrak{G}_1, \mathfrak{G}_2 \subset \mathfrak{G}$ such that $\mathfrak{G}_1 \neq \mathfrak{G}_2$. We may suppose that there is a $G_0 \in \mathfrak{G}_1$ with $G_0 \notin \mathfrak{G}_2$. Because $\mathfrak{G}$ is entwined $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G} \backslash \{G_0\}\}$ is not finer than $\mathscr{T}_{G_0}$. $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G} \backslash \{G_0\}\}$ is finer than $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_2\}$ and $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_1\}$ is finer than $\mathscr{T}_{G_0}$. Thus, $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_2\}$ is not finer than $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_1\}$. □

Now we want to show that there is an entwined $\mathfrak{G}$ of power $2^{\aleph_0}$. We identify each natural number with the set of its predecessors. ${}^n 2$ denotes the set of functions from n into 2 and ${}^\omega 2$ the set of those from ω into 2. If f is such a function, then $f \upharpoonright n$ is the restriction of f to n. By induction over n, we shall choose, for each $f \in {}^{n+1}2$, a condition p^f such that:

(1) $p^{f|n} \leqq p^f$.

(2) $p^f(0, 1) \cap \bigcap \{p^g(n, 0) | g \in {}^{n+1}2 \text{ and } g \neq f\}$ is not empty.

(3) There is an $r \in \bigcap \{p^g(n, 0) | g \in {}^{n+1}2\}$ and an $m \in \omega$, $m \neq 0$, such that $r^m \in \bigcap \{p^g(0, 1) | g \in {}^{n+1}2\}$.

Let $p^\phi = p^0$. Assume that, for $f \in {}^n 2$, p^f is chosen. Let $(f_k)_{k \in m_n}$ be a well ordering of ${}^{n+1}2$. For every $f \in {}^{n+1}2$ we choose, by induction over $k \in m_n + 1$ conditions, p_k^f as follows: By Corollary 8 there are conditions p_*^f such that $p_*^f \geqq p^{f|n}$ and $M = \bigcap \{p_*^f(n, 0) | f \in {}^{n+1}2\} \backslash \{0\} \neq \varnothing$. Let $r \in M$ be given. If there is an $m \in \omega$ such that $r^m = 1$, then take p_0^f to be p_*^f. If there is no $m \in \omega$ such that $r^m = 1$, then $\{r^m | m \in \omega\}$ is infinite. Hence, by Theorem 4 there are conditions $p_0^f \geqq p_*^f$ and an $m \in \omega$ such that $r^m \in \bigcap \{p_0^f(0, 1) | f \in {}^{n+1}2\}$.

Suppose p_k^f is already chosen. Then by Theorem 4 and Corollary 8 there are conditions p_{k+1}^f such that $p_{k+1}^f \geqq p_k^f$ and $p_{k+1}^{f_k}(0, 1) \cap \bigcap \{p_{k+1}^f(n, 0) | f \in {}^{n+1}2 \text{ and } f \neq f_k\}$ is nonempty. Let $p^f = p_{m_n}^f$. Then the conditions p^f, $f \in {}^{n+1}2$, have the desired properties. Now for $g \in {}^\omega 2$, define G_g to be the chain $\{p^{g \upharpoonright n} | n \in \omega\}$, and define $\mathfrak{G}$ to be $\{G_g | g \in {}^\omega 2\}$.

12. THEOREM. *$\mathfrak{G}$ is an entwined set of power $2^{\aleph_0}$ of chains of conditions.*

PROOF. It is sufficient to show that $\mathfrak{G}$ is entwined. Let $g_0 \in {}^\omega 2$ and let $g_i \in {}^\omega 2 \backslash \{g_0\}$, $1 \leqq i \leqq m$. Then for each n there is a $k > n$ such that $g_0 \upharpoonright k \notin \{g_i \upharpoonright k | 1 \leqq i \leqq m\}$. By 2, we know that $p^{g_0 \upharpoonright k}(0, 1) \cap \bigcap \{p^{g_i \upharpoonright k}(k-1, 0) | 1 \leqq i \leqq m\}$ is not empty. Since $k - 1 \geqq n$ we have that $p^{g_i \upharpoonright k}(k-1, 0)$ is a subset of $p^{g_i \upharpoonright k}(n, 0)$. This implies that $p^{g_0 \upharpoonright k}(0, 1) \cap \bigcap \{p^{g_i \upharpoonright k}(n, 0) | 1 \leqq i \leqq m\}$ is not empty. Thus, $\mathfrak{G}$ is entwined. □

Now we shall prove that we have constructed $\mathfrak{G}$ in such a way that for each $\mathfrak{G}_1 \subset \mathfrak{G}$ the topology $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}_1\}$ is not the join of locally bounded ring topologies. By [**1**] it is sufficient to show that there is a neighborhood V of zero such that for every neighborhood $U \subset V$ there is an $n \in \omega$ with $U^n \not\subset V$.

13. THEOREM. *For every* $\mathfrak{G}' \subset \mathfrak{G}$, $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}'\}$ *fails to be the join of locally bounded ring topologies.*

PROOF. Let $G_0 \in \mathfrak{G}'$ be given and let U be a neighborhood of zero, $U \subset V_0^{G_0}$. Then there are finitely many $G_i \in \mathfrak{G}'$, $1 \leqq i \leqq m$, and a $k \in \omega$ such that $\bigcap \{V_k^{G_i} 1 \leqq i \leqq m\} \subset U$. By the definition of $\mathfrak{G}$ there are functions $f_i \in {}^{k+1}2$, $0 \leqq i \leqq m$, such that $p^{f_i}(k, 0) \subset V_k^{G_i}$. From 3 it follows that there is an $r \in \bigcap \{p^{f_i}(k, 0) | 1 \leqq i \leqq m\}$ and a $z \in \omega$, $z \neq 0$, such that $r^z \in p^{f_0}(0, 1)$. Hence, $U^z \not\subset V_0^{G_0}$ and therefore $\bigvee \{\mathscr{T}_G | G \in \mathfrak{G}'\}$ is not the join of locally bounded ring topologies.

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