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TRANSCENDENTAL EXTENSIONS OF FIELD TOPOLOGIES ON COUNTABLE FIELDS

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ABSTRACT. Let $\mathscr{T}$ be a field topology on a countable field K and let $K(x)$ be a simple transcendental extension of K. Then there exists a field topology $\mathscr{T}'$ for $K(x)$ such that $\mathscr{T}' \upharpoonright K = \mathscr{T}$.

Let K be a countable field and let $\mathfrak{A}$ be a fundamental system of neighborhoods at zero. We identify K with the field of constant functions on K and $K(x)$ with the field $R(K)$ of rational functions over K. If D is a nonprincipal filter on K, then we define for each $U \in \mathfrak{A}$, U^D to be the set of all $f \in R(K)$ with $\{r \mid f(r) \in U\} \in D$ and $\mathfrak{A}^D$ to be the set $\{U^D \mid U \in \mathfrak{A}\}$. $\mathfrak{A}^D \upharpoonright K = \mathfrak{A}$, and $\mathfrak{A}^D$ is a filter base. But $\mathfrak{A}^D$ is not necessarily a fundamental system of neighborhoods at zero of a field topology on $R(K)$.

1. DEFINITION. A filter D on K is called $\mathfrak{A}$-generic, if for each $U \in \mathfrak{A}$ and for each $f \in R(K)$ there is a $V \in \mathfrak{A}$ such that $\{r \mid f(r)V \subset U\} \in D$.

2. THEOREM. *If D is a $\mathfrak{A}$-generic filter on K, then $\mathfrak{A}^D$ is a fundamental system on $R(K)$ such that $\mathfrak{A}^D \upharpoonright K = \mathfrak{A}$.*

PROOF. To see that $\mathfrak{A}^D$ defines a group topology on $R(K)$, let $U^D \in \mathfrak{A}^D$ be given. Since $\mathfrak{A}$ is a fundamental system there is a V such that $V - V \subset U$. Suppose $f, g \in V^D$. Then $\{r \mid f(r) \in V\} \in D$ and $\{r \mid g(r) \in V\} \in D$. By

$$\{r \mid f(r) - g(r) \in U\} \supset \{r \mid f(r) \in V\} \cap \{r \mid g(r) \in V\}$$

we have that $f - g \in U^D$. Thus $V^D - V^D \subset U^D$.

By a similar argument, it can be seen that inversion and multiplication at zero are continuous. So it remains to show that multiplication is continuous everywhere. Let $f \in R(K)$ and $U^D \in \mathfrak{A}^D$ be given. Since D is $\mathfrak{A}$-generic there is a V such that $\{r \mid f(r) \cdot V \subset U\} \in D$. Suppose $g \in V^D$. Then $\{r \mid g(r) \in V\} \in D$.

$$\{r \mid g(r) \cdot f(r) \in U\} \supset \{r \mid f(r) \cdot V \subset U\} \cap \{r \mid g(r) \in V\}$$

and therefore $f \cdot g \in U^D$. Thus, multiplication is continuous everywhere. □

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Under certain conditions one can show that there exists a $\mathfrak{A}$-generic filter on K.

3. THEOREM. *If $\mathfrak{A}$ is a countable fundamental system, then there is a $\mathfrak{A}$-generic filter D on K.*

PROOF. If $\{0\} \in \mathfrak{A}$, then take D to be the cofinite subsets of K. Otherwise let $(f_n)_{n\in\omega}$ be a well ordering of $R(K)$, $(U_m)_{m\in\omega}$ a well ordering of $\mathfrak{A}$, and $((n_\alpha, m_\alpha))_{\alpha\in\omega}$ a well ordering of $\omega\times\omega$. By induction over α we choose nonempty open sets 0_α as follows:

Let $0_0=K$. Assume that 0_α has been chosen. Then there is a $r_\alpha\in 0_\alpha$ and a $V_\alpha\in\mathfrak{A}$, such that $f_{n_\alpha}(r_\alpha)$ is defined and $(f_{n_\alpha}(r_\alpha)+V_\alpha)V_\alpha\subset U_{m_\alpha}$. Because f_{n_α} is continuous at r_α, there exists a $V'_\alpha\in\mathfrak{A}$ such that $f_{n_\alpha}(r_\alpha+V'_\alpha)\subset f_{n_\alpha}(r_\alpha)+V_\alpha$. Define $0_{\alpha+1}$ to be $0_\alpha\cap(r_\alpha+V'_\alpha)$. Thus 0_α has been chosen for every $\alpha\in\omega$. By a straightforward argument, it can be seen that $\{0_\alpha | \alpha\in\omega\}$ is a base of a $\mathfrak{A}$-generic filter. □

Thus we have shown that every field topology with countable basis on K, can be extended to a field topology on $K(x)$. Moreover, if the topology on K is locally bounded, then the topology on $K(x)$ is locally bounded. To prove that every field topology on K can be extended to a field topology on $K(x)$ we need the following Lemma:

4. LEMMA. *If $\mathfrak{A}$ is a fundamental system on K and $U\in\mathfrak{A}$, then there is a countable fundamental system $\mathfrak{A}'\subset\mathfrak{A}$ for a field topology with $U\in\mathfrak{A}'$.*

PROOF. Let ϕ_K denote the set of functions φ_d, φ_c, φ_b, and φ_a, $a\in K$, defined by $\varphi_d(V)=V/(1-V)$, $\varphi_c(V)=V\cdot V$, $\varphi_b(V)=V-V$ and $\varphi_a(V)=aV$, for every subset V of K, $1\notin V$. Let $(\varphi_n)_{n\in\omega}$ be a well ordering of ϕ_K and $((n_\alpha, m_\alpha))_{\alpha\in\omega}$ a well ordering of $\omega\times\omega$ such that $\alpha\geqq m_\alpha$.

By induction over α we choose sets $U_\alpha\in\mathfrak{A}$. Let $U_0=U$. Assume that $U_0,\cdots,U_\alpha$ have been chosen from $\mathfrak{A}$. Because $\mathfrak{A}$ is a fundamental system, there is a $V\in\mathfrak{A}$ such that $\varphi_{n_\alpha}(V)\subset U_{m_\alpha}$ and $V\subset U_\alpha$. Let $U_{\alpha+1}$ be such a V. Thus U_α is well defined for $\alpha\in\omega$. It is straightforward to see that $\{U_\alpha | \alpha\in\omega\}$ is a fundamental system, which contains U and is a subset of $\mathfrak{A}$.

5. THEOREM. *If $\mathscr{T}$ is a field topology on K, then there is a field topology $\mathscr{T}'$ on $K(x)$ such that $\mathscr{T}'\upharpoonright K=\mathscr{T}$.*

PROOF. Let $\mathfrak{A}$ be a fundamental system of neighborhoods at zero of $\mathscr{T}$. By Lemma 4 there is a set $\{\mathfrak{A}_U | U\in\mathfrak{A}\}$ of countable fundamental systems such that $\bigcup\{\mathfrak{A}_U | U\in\mathfrak{A}\}=\mathfrak{A}$. From Theorem 3, we see that for each $U\in\mathfrak{A}$ there is an $\mathfrak{A}_U$-generic filter D_U. By Theorem 2 we have that $\mathfrak{A}_U^{D_U}$ is a fundamental system on $K(x)$ such that $\mathfrak{A}_U^{D_U}\upharpoonright K=\mathfrak{A}_U$. Hence, $\bigcup\{\mathfrak{A}_U^{D_U} | U\in\mathfrak{A}\}$ is a subbase for a fundamental system $\mathfrak{A}^*$ on $K(x)$ such that $\mathfrak{A}^*\rceil K=\mathfrak{A}$. If we take $\mathscr{T}'$ to be the field topology defined by $\mathfrak{A}^*$, then $\mathscr{T}'$ has the desired properties.

Reference

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