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ON THE RADIUS OF β-CONVEXITY OF STARLIKE FUNCTIONS OF ORDER α

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ABSTRACT. A function $f(z)=z+a_2z^2+\cdots$ is called β-convex if $f(z)f'(z)/z\neq 0$ in D: $|z|<1$ and if

$$\operatorname{Re}\{(1-\beta)zf'(z)/f(z)+\beta(1+zf''(z)/f'(z))\}>0$$

for some $\beta\geqq 0$ and all z in D. Recently M. O. Reade and P. T. Mocanu have announced a sharp result about the radius of β-convexity for starlike functions. The author generalizes this result to starlike functions of order α.

1. **Introduction.** Let $f(z)=z+a_2z^2+\cdots$ be analytic in the unit disc $D:|z|<1$. We say that $f(z)$ is starlike of order α, $0\leqq\alpha<1$, if

$$(1)\qquad \operatorname{Re}\{zf'(z)/f(z)\}>\alpha$$

for all z in D. We denote such a class of functions by S^*_α. We say that $f(z)$ is convex of order α, $0\leqq\alpha<1$, if

$$(2)\qquad \operatorname{Re}\{1+zf''(z)/f'(z)\}>\alpha,$$

for all z in D. We denote such a class of functions by C_α. For $\alpha=0$, S^*_0, C_0 are simply called starlike and convex, respectively.

We consider now a class of functions which is formed by a linear combination of the conditions stated in (1) and (2).

DEFINITION. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in D with $f(z)f'(z)/z\neq 0$ in D. Let*

$$L(\beta;f)=(1-\beta)zf'(z)/f(z)+\beta(1+zf''(z)/f'(z)).$$

If

$$(3)\qquad \operatorname{Re}\{L(\beta;f)\}>0$$

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for some β, $\beta \geqq 0$, $z \in D$, then $f(z)$ is called a β-convex function. We denote this class by $C(\beta)$.

Mocanu [2] was the first to introduce the class of β-convex functions under the restrictions $0 \leqq \beta \leqq 1$ and that $f(z)$ must be univalent in D. Recently, however, Mocanu and Reade [3] have shown that each function in $C(\beta)$ is univalent (starlike) for $\beta \geqq 0$. In particular each $f(z) \in C(\beta)$ is convex if $\beta \geqq 1$. It is natural now to raise the following question: What is the largest $r_{\alpha,\beta}$, $0 < r_{\alpha,\beta} \leqq 1$ such that each $f(z) \in S_\alpha^*$ is a function in $C(\beta)$ for $|z| < r_{\alpha,\beta}$? Again, Reade and Mocanu [4] have announced a sharp result for the general class S_0^*.

THEOREM A (READE AND MOCANU). *If $f(z) \in S_0^*$, then $f(z) \in C(\beta)$ for $|z| < r_\beta = (1+\beta+((1+\beta)^2-1)^{-1})^{1/2}$, $\beta \geqq 0$. This result is sharp for $f(z) = z/(1-z)^2$.*

We call $r_{\alpha,\beta}$ the radius of β-convexity of the class S_α^*. Here $r_{0,\beta} = r_\beta$.

The object of this note is to extend Theorem A to the class S_α^*; in short to find $r_{\alpha,\beta}$. In §2 a rough estimate of $r_{\alpha,\beta}$ is given, Theorem 1. In §3, the number $r_{\alpha,\beta}$ is completely determined, Theorem 2. The method used in §3 is that of V. A. Zmorovič [7]. We also adopt his notations and thus we refer the reader to [7] for a deeper and perhaps a better understanding of §3.

2. **Some estimate for $r_{\alpha,\beta}$.** Let P be the class of analytic functions in D such that if $p(z) \in P$, $p(0)=1$, and $\mathrm{Re}\{p(z)\} > 0$ for all $z \in D$. Let $q(z) = zf'(z)/f(z)$, where $f(z) \in S_\alpha^*$. Then there exists $p(z) \in P$ such that

$$q(z) = \alpha + (1-\alpha)p(z) = (p(z)+h)/(1+h), \tag{4}$$

where $h = \alpha/(1-\alpha)$.

Using (3), (4) and the fact that

$$1 + zf''(z)/f'(z) = q(z) + zq'(z)/q(z),$$

the radius of β-convexity of the class S_α^*, $r_{\alpha,\beta}$ becomes the smallest positive root of $Q_{\alpha,\beta}(r) = 0$, where

$$Q_{\alpha,\beta}(r) = \min_{p \in P} \min_{|z|=r<1} \mathrm{Re}\{(1-\alpha)(p(z)+h) + \beta zp'(z)/(p(z)+h)\}. \tag{5}$$

Thus our problem is now reduced to finding the quantity

$$Q(r) = \min_{p \in P} \min_{|z|=r<1} \mathrm{Re}\{\Psi(p(z), zp'(z))\}, \tag{6}$$

where $\Psi(w, W)$ is an analytic function of the variables w and W in the W-plane and in the half plane $\mathrm{Re}\{w\} > 0$. It is known [5] that the minimum in (6) is realized for functions of the form

$$p(z) = \lambda_1 \frac{1 + ze^{-i\theta_1}}{1 - ze^{-i\theta_1}} + \lambda_2 \frac{1 + ze^{-i\theta_2}}{1 - ze^{-i\theta_2}}, \tag{7}$$

where $\theta_1, \theta_2 \in [0, 2\pi]$, $\lambda_1, \lambda_2 \geqq 0$, and $\lambda_1 + \lambda_2 = 1$.

Before determining $Q(r)$, an elementary method yields a rough estimate for $r_{\alpha,\beta}$. We need the following lemmas.

LEMMA 1. *If $\phi(z)$ is analytic and $|\phi(z)| \leqq 1$ in D, then*

$$|\phi'(z)| \leqq (1 - |\phi(z)|^2)/(1 - r^2)$$

for $|z| = r < 1$.

Lemma 1 may be found in Carathéodory [**1**, p. 18].

LEMMA 2. *If $q(z) = 1 + b_1 z + \cdots$, $\operatorname{Re}\{q(z)\} > \alpha$, then*

$$\operatorname{Re}\{q(z)\} \geqq (1 - (1 - 2\alpha)r)/(1 + r)$$

valid for $|z| = r < 1$.

This estimate readily follows from the fact that

$$q(z) = (1 - (1 - 2\alpha)z\phi(z))/(1 + z\phi(z)),$$

where $\phi(z)$ is as in Lemma 1.

LEMMA 3. *If $q(z)$ is as in Lemma 2, then*

$$|zq'(z)/q(z)| \leqq 2(1 - \alpha)r/(1 - r)(1 + (1 - 2\alpha)r), \tag{8}$$

for $|z| = r < 1$.

PROOF. From

$$q(z) = (1 - (1 - 2\alpha)z\phi(z))/(1 + z\phi(z)),$$

it follows that

$$zq'(z)/q(z) = \frac{-2(1 - \alpha)(z^2\phi'(z) + z\phi(z))}{(1 + z\phi(z))(1 - (1 - 2\alpha)z\phi(z))}.$$

The above may be written in the form

$$zq'(z)/q(z) = -2(1 - \alpha)I_1(z)I_2(z), \tag{9}$$

where

$$I_1(z) = (z^2\phi'(z) + z\phi(z))/(1 - z^2\phi^2(z)),$$
$$I_2(z) = (1 - z\phi(z))/(1 - (1 - 2\alpha)z\phi(z)).$$

Using the triangular inequality, the monotonicity of the right-hand side of $|I_1(z)|$ with respect to $|\phi'(z)|$ and on applying Lemma 1, we get

$$|I_1(z)| \leqq \frac{2}{1-r^2}\left(\frac{r^2(1-|\phi(z)|^2)+r\,|\phi(z)|\,(1-r^2)}{1-r^2\,|\phi(z)|^2}\right).$$

Let $0 \leqq |\phi(z)| = t \leqq 1$. The above inequality becomes

$$|I_1(z)| \leqq g(t,r) = \frac{2}{1-r^2}\left(\frac{r^2(1-t^2)+rt(1-r^2)}{1-r^2t^2}\right).$$

For fixed r,

$$\partial g/\partial t = 2r(1-rt)^2/(1-r^2t^2)^2 \geqq 0.$$

Therefore,

$$|I_1(z)| \leqq \max_{0\leqq t\leqq 1} g(t,r) = g(1,r) = 2r/(1-r^2).$$

It is also clear that

$$|I_2(z)| \leqq (1+r)/(1+(1-2\alpha)r).$$

Using these estimates in (9) one gets (8).

THEOREM 1. *Let $f(z)=z+a_2z^2+\cdots$ be a function in S^*_α. Then $f(z)$ is β-convex in $|z|<R_{\alpha,\beta}$, where $R_{\alpha,\beta}$ is the smallest positive root of*

$$(10) \qquad \begin{aligned}(1-2\alpha)^2r^3 - ((1-2\alpha)^2 + 2\beta(1-\alpha))r^2 \\ - (1+2\beta(1-\alpha))r + 1 = 0.\end{aligned}$$

PROOF. Let $q(z)=zf'(z)/f(z)$. From (4),

$$\begin{aligned}&\mathrm{Re}\{(1-\alpha)(p+h) + \beta zp'(z)/(p(z)+h)\} \\ &\qquad = \mathrm{Re}\{q(z) + \beta zq'(z)/q(z)\} \leqq \mathrm{Re}\{q(z)\} - \beta\,|zq'(z)/q(z)|,\end{aligned}$$

where $h=\alpha/(1-\alpha)$, $0\leqq\alpha<1$.

Applying Lemmas 2 and 3,

$$\begin{aligned}&\mathrm{Re}\{(1-\alpha)(p(z)+h) + \beta zp'(z)/(p(z)+h)\} \\ &\leqq [(1-2\alpha)^2r^3 - ((1-2\alpha)^2 + 2\beta(1-\alpha))r^2 - (1+2\beta(1-\alpha))r + 1] \\ &\qquad \div (1-r^2)(1+(1-2\alpha)r).\end{aligned}$$

From (5) and the above inequality each starlike function of order α in D is β-convex in $|z|<R_{\alpha,\beta}$, where $R_{\alpha,\beta}$ is given by (10). Note that $r_{\alpha,\beta}\geqq R_{\alpha,\beta}$ and if $\alpha=0$, $r_{0,\beta}=R_{0,\beta}=r_\beta$ as given in Theorem A. Theorem 1, however, is not sharp since the estimates of Lemmas 2, 3 are sharp for

$$q_\alpha(z) = (1-(1-2\alpha)z)/(1+z)$$

but not at the same point. Indeed, $q_\alpha(z)$ realizes the estimate of Lemma 2 at $z=r$, while realizing the estimate of Lemma 3 at $z=-r$. This is precisely the source of difficulties in such extremal problems.

3. The main theorem for $r_{\alpha,\beta}$. In this section we obtain $r_{\alpha,\beta}$ through an application of a theorem and technique due to V. A. Zmorovič [7] which is stated next.

THEOREM B (V. A. ZMOROVIČ). *Let* $\Psi(w, W)=M(w)+N(w)W$, *where* $M(w)$ *and* $N(w)$ *are defined and are finite in the half plane* $\text{Re}\{w\}>0$. *Set*

$$w = \lambda_1 \frac{1+z_1^m}{1-z_1^m} + \lambda_2 \frac{1+z_2^m}{1-z_2^m},$$

$$W = \lambda_1 \frac{2mz_1^m}{(1-z_1^m)^2} + \lambda_2 \frac{2mz_2^m}{(1-z_2^m)^2},$$

where z_1 *and* z_2 *are any points on* $|z|=r<1$, $\lambda_1\geqq 0$, $\lambda_2\geqq 0$, $\lambda_1+\lambda_2=1$. *Then* $\Psi(w, W)$ *can be put in the form*

$$\Psi(w, W) = M(w) + \frac{m}{2}(w^2-1)N(w) + \frac{m}{2}(\rho^2-\rho_0^2)N(w)e^{2i\psi}$$

where

$$(1+z_k^m)/(1-z_k^m) = a + \rho e^{i\psi_k} \qquad (k=1, 2),$$

$$w = a + \rho_0 e^{i\psi_0}, \qquad 0 \leqq \rho_0 \leqq \rho,$$

$$a = \frac{1+r^{2m}}{1-r^{2m}}, \quad \rho = \frac{2r^m}{1-r^{2m}}, \quad e^{i\psi} = ie^{i(\psi_1+\psi_2)/2}.$$

Also

$$\min \text{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w)$$

(11) $$= \text{Re}\left\{M(w) + \frac{m}{2}(w^2-1)N(w)\right\} - \frac{m}{2}|N(w)|(\rho^2-\rho_0^2).$$

This minimum is reached when

(12) $$\exp[i(2\psi + \arg N(w))] = -1.$$

In our particular problem (5), $m=1$,

$$M(w) = (1-\alpha)(w+h), \qquad N(w) = \beta/(w+h).$$

Thus from (6), (11) and the above relations,

$$\min \text{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w)$$

(13) $$= \text{Re}\left\{(1-\alpha)(w+h) + \frac{\beta}{2}\frac{w^2-1}{w+h}\right\} - \frac{\beta}{2}\frac{\rho^2-\rho_0^2}{|w+h|}.$$

The following remarks will be used later.

REMARK 1. *For a fixed* $w=a+\rho_0 e^{i\psi_0}$, $\rho_0<\rho$, *and a suitably defined* ψ_1 *and* ψ_2, *a choice of* λ_1 *and* λ_2 *may be made, namely,*

$$\lambda_1/\lambda_2 = |\rho e^{i\psi_2} - \rho_0 e^{i\psi_0}|/|\rho e^{i\psi_1} - \rho_0 e^{i\psi_0}|$$

such that $\Psi=(\Psi_1+\Psi_2+\pi)/2$ *becomes the angle of inclination of the secant through* $a+\rho_e^{i\psi_0}$ *and intersects the circle* $|w-a|=\rho$ *at* $a+\rho e^{i\psi_k}$, $k=1,2$. *The choice of* λ_1/λ_2 *is to maintain the correct relations between* $\rho_0 e^{i\psi_0}$, $\rho e^{i\psi_1}$ *and* $\rho e^{i\psi_2}$ *as required in the theorem. Thus* ψ *may assume any value in* $[0, \pi]$. *Also as a consequence of formula* (12), *the minimum in* (13) *is reached when the point* w, $|w-a|<\rho$ *is fixed and the secant through it, as described above, is perpendicular to* $e^{i\phi/2}$, *where* $w+h=Re^{i\phi}$.

If we set $w=a+\xi+i\eta$, $\rho_0^2=\xi^2+\eta^2\leqq\rho^2$, then (13) becomes

$$\Psi_\rho(w) \equiv \Psi_\rho(\xi,\eta) = \left(1+\frac{\beta}{2}-\alpha\right)(a+\xi+h)-\beta h + \frac{\beta}{2}(h^2-1)(a+\xi+h)R^{-2} - \frac{\beta}{2}(\rho^2-\xi^2-\eta^2)R^{-1}, \tag{14}$$

where $R^2=(a+\xi+h)^2+\eta^2$.

One can show that $\partial\Psi_\rho/\partial\eta=(\beta/2)\eta R^{-4}S(\xi,\eta)$, where

$$\begin{aligned} S(\xi;\eta) &= [\xi^2+4(a+h)\xi+\rho^2+\eta^2+2(a+h^2)]R \\ &\quad - [(h^2-1)(\xi+a+h)] \\ &\geqq [\xi^2+4(a+h)\xi+\rho^2+2(a+h)^2-2(h^2-1)](\xi+a+h) \\ &> 0, \end{aligned}$$

which shows that the minimum of $\Psi_\rho(\xi,\eta)$ on every chord ξ-constant is reached when $\eta=0$. Therefore, the minimum of $\Psi_\rho(\xi,\eta)$ in the circle $\xi^2+\eta^2\leqq\rho^2$ is reached on the diameter $\eta=0$. Now set $\eta=0$ and $R=a+\xi+h$ in (1.4), we arrive at the following:

$$\Psi_\rho(\xi,0) \equiv l(R) = \left(1+\frac{\beta}{2}-\alpha\right)(a+\xi+h)-\beta h + \frac{\beta}{2}(h^2-1)(a+\xi+h)R^{-2} - \frac{\beta}{2}(\rho^2-\xi^2)R^{-1}.$$

From $\xi=R-(a+h)$, $\rho^2=a^2-1$,

$$l(R) = (1+\beta-\alpha)R+\beta(h^2+ah)R^{-1}-\beta(a+2h). \tag{15}$$

Thus $Q(r)=\min l(R)$, $R\in[a+h-\rho, a+h+\rho]$, $Q(r)$ is given by (6). Simple calculations show that the absolute minimum of $l(R)$ is realized at

$$R_0 = \left(\frac{\beta(h^2 + ah)}{1 + \beta - \alpha}\right)^{1/2} \tag{16}$$

Since

$$R_0^2 = \frac{\beta(h^2 + ah)}{1 + \beta - \alpha} < h^2 + ah < (a + h + \rho)^2,$$

$R_0 < a+h+\rho$. However, R_0 may not be greater than $a+h-\rho$. Therefore, if $R_0 \notin [a+h-\rho, a+h+\rho]$, then the minimum of $l(R)$ is obtained at

$$R_1 = a + h - \rho. \tag{17}$$

The radius $r_{\alpha,\beta}$ is therefore determined either from

$$Q(r) = \min l(R) = l(R_0) = 0, \tag{18}$$

with R_0 given by (16), or from

$$Q(r) = \min l(R) = l(R_1) = 0, \tag{19}$$

with R_1 given by (17).

These two equations coincide for some α_0 which will be determined later.

Equations (18) and (19) may be written in the form, respectively,

$$\beta a^2 - 4\alpha a - 4\alpha h = 0, \tag{20}$$

$$(1 - 3\alpha + \alpha h)r^2 - 2(1 + \beta - \alpha - \alpha h)r + 1 + \alpha + \alpha h = 0. \tag{21}$$

It follows from (20) that

$$r_1 = r_{\alpha,\beta} = \left(\frac{2\alpha - \beta + 2(\alpha^2 + \alpha h\beta)^{1/2}}{2\alpha + \beta + 2(\alpha^2 + \alpha h\beta)^{1/2}}\right)^{1/2}. \tag{22}$$

Also from (21) follows that

$$\begin{aligned} r_2 = r_{\alpha,\beta} = [(1 - 2\alpha + \beta(1 - \alpha) \\ + ((1 - 2\alpha + \beta(1 - \alpha))^2 - (1 - 2\alpha)^2)^{1/2}]^{-1}. \end{aligned} \tag{23}$$

However, formula (23) cannot be used to determine $r_{\alpha,\beta}$ if

$$\alpha \geqq (-\beta + (\beta^2 + 8\beta)^{1/2})/4, \tag{24}$$

since r_2 would become greater than 1.

Also formula (22) cannot be used to determine $r_{\alpha,\beta}$ if

$$\alpha \leqq \beta/(4 + \beta), \tag{25}$$

since r_1 would become a nonreal number.

To find α_0 that makes the transition from (23) to (22), we set

$$r_1 = r_2, \tag{26}$$

and solve for $\alpha=\alpha_0$ where α_0 is the smallest positive root of (26) which lies in

$$\left(\frac{\beta}{4+\beta}, \frac{-\beta+(\beta^2+8\beta)^{1/2}}{4}\right).$$

Thus we have our main theorem.

THEOREM 2. *Let α_0 be the smallest positive root of* (26) *which lies in the interval*

$$\left(\frac{\beta}{4+\beta}, \frac{-\beta+(\beta^2+8\beta)^{1/2}}{4}\right).$$

Then the radius of β-convexity for the class S_α^* is determined from (22) *when* $\alpha_0 \leqq \alpha < 1$ *and from* (23) *when* $0 \leqq \alpha \leqq \alpha_0$.

Now we determine the extremal functions $f_0(z)$ for Theorem 2. Using Remark 1 and the fact that the minimum in case (22) is reached at a point on the diameter $\eta=0$ (not an endpoint) one gets $\psi_1 \equiv -\psi_2 \pmod{2\pi}$, and $\lambda_1/\lambda_2=1$. The extremal function given by (7) is therefore of the form

$$p(z) = \frac{1}{2}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1}{2}\frac{1+ze^{i\theta}}{1-ze^{i\theta}},$$

where θ is given by

$$R_0 = \operatorname{Re}\{h+w\} = h + (1-r_1^2)(1-2r_1\cos\theta + r_1^2)^{-1}, \tag{27}$$

r_1 is given by (22) and R_0 is given by (16). Hence the extremal function

$$f_0(z) = z(1-2z\cos\theta + z)^{-1+\alpha}. \tag{28}$$

In case of (22), the minimum is realized at an end point of the diameter $\eta=0$, thus $\psi_1 \equiv \psi_2 \pmod{2\pi}$. The function $p(z)$ of (7) has the form

$$p(z) = (1+ze^{-i\theta})/(1-ze^{-i\theta}),$$

or simply $p(z)=(1+z)/(1-z)$. Hence the extremal function

$$f_0(z) = z(1-z)^{-2(1-\alpha)}. \tag{29}$$

REMARK 2. (i) *The case $\beta=1$ reduces $r_{\alpha,\beta}$ to be the radius of convexity of the class S_α^* which has been previously known for $\alpha=0$, $\alpha=\frac{1}{2}$. V. A. Zmorovič has provided the complete solution for such a case.*

(ii) *A. Schild* [6] *attempted to solve this problem* ($\beta=1$) *and actually succeeded if a certain condition is to be true. In fact he obtained $r_{\alpha,1}$ as* *given by* (22) *and* (23). *Schild also calculated* $\alpha_0=0.335\cdots$ *and found the extremal functions* (28) *and* (29) (*for the case* $\beta=1$).

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REFERENCES

1. C. Carathéodory, *Funktionentheorie.* Band 2, Birkhäuser, Basel, 1950; English transl., *Theory of functions of a complex variable.* Vol. 2, Chelsea, New York, 1954. MR **12**, 248; MR **16**, 346.

2. P. T. Mocanu, *Une propriété de convexité généralisée dans la représentation conforme*, Mathematica (Cluj) **11** (1969), 127–133. MR **42** #7881.

3. P. T. Mocanu and M. O. Reade, *The order of starlikeness of certain univalent functions*, Notices Amer. Math. Soc. **18** (1971), 815. Abstract #71T–B182.

4. M. O. Reade and P. T. Mocanu, *The radius of* α*-convexity of starlike functions*, Notices Amer. Math. Soc. **19** (1972), A-110. Abstract #691-30-1.

5. M. S. Robertson, *Variational methods for functions with positive real part*, Trans. Amer. Math. Soc. **102** (1962), 82–93. MR **24** #A3288.

6. A. Schild, *On starlike functions of order* α, Amer. J. Math. **87** (1965), 65–70. MR **30** #4929.

7. V. A. Zmorovič, *On bounds of convexity for starlike functions of order* α *in the circle* $|z|<1$ *and the circular region* $0<|z|<1$, Mat. Sb. **68 (110)** (1965), 518–526; English transl., Amer. Math. Soc. Transl. (2) **80** (1969), 203–213. MR **33** #5875.

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