ON A BERGMAN-WHITTAKER TYPE OPERATOR IN FIVE OR MORE VARIABLES¹

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ABSTRACT. It is shown that there cannot exist a simple generating function for homogeneous harmonic polynomials in five or more variables similar to those known to exist for three and four variables. Thus there is no simple immediate generalization of the three dimensional Bergman-Whittaker operator (and Gilbert's four dimensional operator) to five or more dimensions.

The classical Bergman-Whittaker integral operator [5, p. 50] maps analytic functions of two complex variables onto solutions of,

$$(1) \quad \Delta_n u = 0$$

where $n=3$ and Δ_n represents the n -dimensional Laplacian. Considerable work has been undertaken to generalize this operator to more general equations (see for example the paper by Colton [1]). To a large degree the success of the techniques involved hinges upon the existence of a certain "auxiliary variable,"

$$(2) \quad \mu = x_1 + \zeta(x_2 + ix_3) + \zeta^{-1}(-x_2 + ix_3).$$

This auxiliary variable has the property [3, p. 540] that if one sets

$$(3) \quad \mu^N = \sum_{i=-N}^N \zeta^i h_{N,i}(x_1, x_2, x_3), \quad N = 1, 2, \dots,$$

then the set, $\{h_{N,i}\}$, forms a maximal linearly independent set of N th degree homogeneous harmonic polynomials in the variables x_1, x_2 , and x_3 .

For the four dimensional case ($n=4$) of equation (1) Gilbert [5, p. 75] has devised the auxiliary variable,

$$(4) \quad \mu = (x_1 + ix_2) + \eta^{-1}\zeta^{-1}(x_1 - ix_2) + \zeta^{-1}(x_3 + ix_4) + \eta^{-1}(-x_3 + ix_4)$$

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which has the property that if one sets

$$(5) \quad \mu^N = \sum_{i=0}^N \sum_{j=0}^N \eta^{-i} \zeta^{-j} h_{N,i,j}(x_1, x_2, x_3, x_4)$$

then the $\{h_{N,i,j}\}$ form a maximal linearly independent set of N th degree homogeneous harmonic polynomials in x_1, x_2, x_3 , and x_4 . Gilbert's auxiliary variable has been used extensively for developing integral operator solutions of four dimensional elliptic equations (see for example [2] or [6, p. 92]).

It is clear that if such auxiliary variables could be developed for higher dimensions then much of the work already accomplished could be easily extended to higher dimensional elliptic equations. Attempts have been made (see for example [5, p. 82] or [6, p. 164]) to develop such higher dimensional auxiliary variables, but they have not been entirely satisfactory.

We show in this paper that there does not exist such an auxiliary variable for dimensions $n \geq 5$, so that the techniques such as those employed by Bergman, Gilbert, and Colton cannot be extended to higher dimensions without major modifications.

THEOREM. *There is no variable, μ , which, in the sense of Bergman-Whittaker (for $n=3$) and Gilbert (for $n=4$), generates the homogeneous linearly independent harmonic polynomials in $n \geq 5$ variables.*

PROOF. Suppose on the contrary that there exists

$$(6) \quad \mu = A_1 N_1 + \cdots + A_n N_n$$

where the set $\{A_j\}$ is a set of nontrivial linearly independent homogeneous harmonic polynomials of degree one in the variables $x_1, \cdots, x_n$, and the $\{N_i\}$ are independent products (sums are not allowable if we are to have a μ "in the sense of Bergman-Whittaker") of $\zeta_1, \zeta_2, \cdots, \zeta_p$ and/or $1/\zeta_1, \cdots, 1/\zeta_p$ ($n=p+2$). We must have that μ^m for $m \geq 0$ is harmonic and in particular μ^2 is harmonic,

$$(7) \quad \mu^2 = A_1^2 N_1^2 + 2A_1 A_2 N_1 N_2 + \cdots + A_n^2 N_n^2.$$

In equation (7) there can be at most

$$(8) \quad n + (n-1) + \cdots + 3 + 2 + 1 = n(n+1)/2$$

distinct terms. But there are

$$(9) \quad h(2, p) = h(2, n-2) = (n-1)(n+2)/2$$

homogeneous linearly independent harmonic polynomials of degree two

in n variables [4, V.II, p. 237]. And

$$(10) \quad n(n+1)/2 - (n-1)(n+2)/2 = 1.$$

So in order to have the proper number of polynomials of degree two as coefficients of the $N_i N_j$ in μ^2 , exactly two terms must combine, no more and no less (i.e. $N_i N_j = N_l N_m$ for $i \neq l, m$ and $j \neq l, m$ once and only once).

Since we started with ($n \geq 5$) distinct A_j , at least one term, which we can without loss of generality take to be A_1 , will be such that the set $\{A_1 A_j | j=1, \dots, n\}$ is a set of n homogeneous linearly independent harmonic polynomials of degree two, i.e. each member of the set will appear as a distinct coefficient in μ^2 . Let $A_j = a_{j,1}x_1 + a_{j,2}x_2 + \dots + a_{j,n}x_n$. Since the members of the set $\{A_j\}$ are nontrivial and linearly independent, the n vectors of the set, $\{(a_{j,1}, a_{j,2}, \dots, a_{j,n}) : j=1, \dots, n\}$ are a basis for C^n . But the condition that $A_1 A_j$ is harmonic for $j=1, \dots, n$ is equivalent to the condition that the inner products vanish,

$$\langle (\bar{a}_{1,1}, \dots, \bar{a}_{1,n}), (a_{j,1}, \dots, a_{j,n}) \rangle = 0$$

for $j=1, \dots, n$. This means that the vector, $(\bar{a}_{1,1}, \dots, \bar{a}_{1,n})$, is orthogonal to every member of a basis for C^n . It follows that $A_1 \equiv 0$ which is a contradiction.

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