L^2 -SOLUTIONS TO $y'' + c(t)y' + a(t)b(y) = 0$

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ABSTRACT. Two theorems are presented giving sufficient conditions for all solutions to $y'' + c(t)y' + a(t)b(y) = 0$ to be bounded and square integrable over the nonnegative real line.

In this paper, sufficient conditions are given for all solutions to the differential equation

$$(1) \quad y'' + c(t)y' + a(t)b(y) = 0$$

to be square integrable over $[0, \infty)$. In proving this, we shall show that the derivatives are also square integrable over the same interval. We now state and prove our first result.

THEOREM I. *Suppose $c(t)$, $a(t)$ and $b(y)$ are all continuous functions. Furthermore, let $c(t) > \alpha > 0$, $a(t) > \alpha$, and $a'(t) \leq 0$ on $[0, \infty)$, and $\lim_{|y| \rightarrow \infty} B(y) = \int_0^y b(u) du = \infty$, then all solutions to (1) are bounded as $t \rightarrow \infty$ and $\int_0^\infty y'^2 dt < \infty$.*

PROOF. Multiply (1) by $2y'$ to get

$$(2) \quad 2y'y'' + 2c(t)y'^2 + 2a(t)b(y)y' = 0.$$

Integrating by parts from 0 to t we have

$$(3) \quad y'(t)^2 + 2 \int_0^t c(s)y'^2(s) ds + 2a(t)B(y(t)) - 2 \int_0^t a'(s)B(y(s)) ds = K$$

where $K = y'(0)^2 + 2a(0)B(y(0))$. The above implies $|y|$ remains bounded as $t \rightarrow \infty$ and $\int_0^\infty y'^2 dt < \infty$. Otherwise, the left side would become infinite which is impossible.

We now give sufficient conditions for the solutions to be L^2 -solutions.

THEOREM II. *The hypotheses are the same as Theorem I. In addition, if $c'(t) \leq 0$ on $[0, \infty)$ and $yb(y) \geq \beta y^2$ where $\beta > 0$, then $\int_0^\infty y^2 dt < \infty$.*

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PROOF. Multiply (1) by y to get

$$(4) \quad yy'' + c(t)yy' + a(t)b(y)y = 0.$$

Integrating by parts from 0 to t we have

$$(5) \quad \begin{aligned} & y(t)y'(t) - \int_0^t y'(s)^2 ds + \frac{y^2(t)}{2} c(t) \\ & - \frac{1}{2} \int_0^t y^2(s) \frac{dc}{ds} ds + \int_0^t a(s)b(y)y ds = y(0)y'(0) + \frac{y^2(0)}{2} c(0). \end{aligned}$$

Letting $t \rightarrow \infty$, we immediately conclude

$$(6) \quad \int_0^\infty y^2 dt < \infty.$$

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