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CHAIN TYPE DECOMPOSITION IN INTEGRAL DOMAINS

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ABSTRACT. Let R be a (skew) integral domain. For $0 \neq a \in R$, a is simple if the interval $[aR, R]$ of principal right ideals of R containing aR is not the union of two proper subintervals of $[aR, R]$. It is shown that each irredundant factorization of an element of R into simple elements is unique up to multiplication by units.

All rings considered are (skew) integral domains, that is, rings with unity without proper zero divisors. Let R be a ring and let R^* be the monoid of its nonzero elements. If $a, b \in R^*$ with $aR \subset bR$, the set $[aR, bR] = \{xR \mid aR \subset xR \subset bR\}$ is partially ordered by inclusion. If $[aR, bR] = [aR, xR] \cup [xR, R]$ we say that xR *splits* $[aR, bR]$. Let $\mathrm{Ch}(a, b) = \{xR \mid xR \text{ splits } [aR, bR]\}$. Clearly $\mathrm{Ch}(a, b)$ is a chain in $[aR, bR]$. We write $\mathrm{Ch}(a)$ for $\mathrm{Ch}(a, 1)$, and this is called the *chain for a*. If $\mathrm{Ch}(a) = \{aR, R\}$, a is said to be (*right*) *simple*; if $\mathrm{Ch}(a) = [aR, R]$, a is said to be (*right*) *rigid*. As we shall show, both of these concepts are left-right symmetric. In this paper we are interested in the decomposition of elements into simple elements and the uniqueness of such decompositions.

We begin with a general statement from which the left-right symmetry of the definitions given above will follow.

PROPOSITION 1. *Let R be a ring and let $a \in R^*$. The posets $[aR, R]$ and $[Ra, R]$ are dually isomorphic; in particular, if $a = xx'$ then the correspondence $xR \leftrightarrow Rx'$ is a bijection which reverses order.*

PROOF. Let $a = xx' = yy'$. If $xR \subset yR$ then $x = yz$ for some z; thus $zx' = y'$ and $Ry' \subset Rx'$. Reversing the argument we have $xR \subset yR$ if $Ry' \subset Rx'$. Note that $xR = yR$ if and only if $Ry' = Rx'$ which is the case if and only if z is a unit.

COROLLARY 2. *An element is right rigid if and only if it is left rigid.*

If $a = xx'$ then from Proposition 1 we see that xR splits $[aR, R]$ if and only if Rx' splits $[Ra, R]$ from which we deduce the following.

COROLLARY 3. *An element is right simple if and only if it is left simple.*

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Recall that a *prime* (also called *atom*, *irreducible*, etc.) is a nonunit in R^* that has no proper factors. An equivalent definition is: a nonunit $p \in R^*$ is prime if $[pR, R]=\{pR, R\}$. This yields the following relationship of prime to simple and rigid elements.

PROPOSITION 4. *A nonunit in R^* is prime if and only if it is both simple and rigid.*

If R has the acc for principal right and principal left ideals then for each $a \in R^*$, $[aR, R]$ has the acc and the dcc by Proposition 1; in particular $\mathrm{Ch}(a)$ is a finite chain. More generally let R be any ring, let $a \in R^*$ and suppose that the chain $\mathrm{Ch}(a)$ is finite, say

$$aR = y_kR \subsetneq y_{k-1}R \subsetneq \cdots \subsetneq y_1R \subsetneq y_0R = R.$$

Then writing $y_{j+1}=y_js_{j+1}$ for $j=0, 1, \cdots, k-1$ (where $y_0=1$) we see that a has the factorization

$$a = s_1s_2 \cdots s_k \tag{1}$$

where each s_j is simple and the subproducts $s_i \cdots s_j$, for $i<j$, are not simple. Such a factorization of a into simple elements is said to be *irredundant*. We may establish the converse as follows.

THEOREM 5. *If $a \in R^*$ has an irredundant factorization $a=s_1s_2 \cdots s_k$ into simple elements s_j then* $\mathrm{Ch}(a)$ *is the finite chain*

$$aR = s_1s_2 \cdots s_kR \subset s_1s_2 \cdots s_{k-1}R \subset \cdots \subset s_1R \subset R.$$

PROOF. Assume $k>1$ and let $z_j=s_1 \cdots s_j$. Since z_k is not simple we may choose xR which properly splits $[z_kR, R]$. Thus $xR\subset z_{k-1}R$ or $z_{k-1}R\subset xR$. In the first case xR must split $[z_kR, z_{k-1}R]$ which is isomorphic (as poset) to $[s_kR, R]$. Since s_k is simple we have $xR=z_{k-1}R$. In the second case xR splits $[z_{k-1}R, R]$ and we repeat the process, eventually arriving at $xR=z_jR$ for some $j<k$. Thus $\mathrm{Ch}(a)=\mathrm{Ch}(z_k, z_j)\cup\mathrm{Ch}(z_j)$. By induction, $\mathrm{Ch}(z_j)$ is the chain

$$z_jR \subset z_{j-1}R \subset \cdots \subset z_1R \subset R \tag{2}$$

and $\mathrm{Ch}(s_{j+1} \cdots s_k)$ is the chain

$$s_{j+1} \cdots s_kR \subset \cdots \subset s_{j+1}R \subset R. \tag{3}$$

Multiplying (3) on the left by z_j we see that $\mathrm{Ch}(z_k, z_j)$ is the chain

$$z_kR \subset \cdots \subset z_jR. \tag{4}$$

Putting (2) and (4) together we obtain the desired form for $\mathrm{Ch}(a)$.

Since two irredundant factorizations of a into simple elements give rise to the same chain for a we have the following result which Johnson first obtained in [2] for the case of principal right ideal domains (cf. the treatment given by P. M. Cohn [1] for 2-firs, i.e. weak Bezout domains).

THEOREM 6. *If $a \in R^*$ has an irredundant factorization into simple elements as in* (1) *then this is unique up to unit factors.*

A *weak valuation ring* is a ring in which $aR \cap bR \neq 0$ implies $aR \subset bR$ or $bR \subset aR$. Thus R is a weak valuation ring if and only if each element of R^* is rigid. According to Proposition 4 the concepts of "simple" and "prime" coincide in this case; thus Theorem 6 has the following form (cf. [3] where weak valuation rings are characterized as [rigid] unique factorization domains).

COROLLARY 7. *Let R be a weak valuation ring. If an element in R^* has a factorization into primes then this is unique up to unit factors.*

If aR has a decomposition $aR = \bigcap_i a_i R$ then this decomposition is said to be *irredundant* if no $a_i R$ can be omitted from the intersection. Elements $a_i \in R^*$ are *left coprime* if there is no nonunit which is a left factor of each a_i. A *complete decomposition* of aR is an irredundant decomposition $aR = \bigcap_i a_i R$ into factors a_i that are left coprime. The following theorem gives a method for finding simple elements in a ring.

THEOREM 8. *If $aR = \bigcap_i a_i R$ is a complete decomposition of aR into at least two factors a_i then a is simple.*

PROOF. Suppose xR splits $[aR, R]$. Since the decomposition is irredundant we must have $xR \subset a_i R$ for each i, in which case $xR = aR$, or else $a_i R \subset xR$ for each i, in which case $xR = R$ since the a_i are left coprime.

We turn briefly to an example. Let R be a subcommutative ($aR = Ra$ for all $a \in R$) unique factorization domain and let $a \in R^*$ have factorization

$$a = p_1^{n_1} \cdots p_k^{n_k} u \tag{5}$$

where p_i are pairwise nonassociated primes and u is a unit in R. Using familiar arguments (5) yields

$$aR = p_1^{n_1} R \cap \cdots \cap p_k^{n_k} R.$$

According to Theorem 8, a is simple if $k > 1$ and a is rigid if $k = 1$. If a is rigid then its irredundant factorization into simple elements has the form $a = p^n u$ where p is a prime and u is a unit.

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