PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

A CLASS OF FLAG TRANSITIVE PLANES

M. L. NARAYANA RAO

ABSTRACT. A class of translation affine planes of order q^2, where q is a power of a prime $p\geqq 3$ is constructed. These planes have an interesting property that their collineation groups are flag transitive.

1. **Introduction.** Let π be a finite affine plane of order n. A collineation group G of π is defined to be flag transitive on π if G is transitive on the incident point-line pairs, or flags, of π. A. Wagner [7] has shown that π is a translation plane so that $n=p^r$ for some prime p and for some integer $r>0$. D. A. Foulser [3], [4] has determined all flag transitive groups of finite affine planes. While determining the flag transitive groups Foulser remarks that the existence of non-Desarguesian flag transitive affine planes is still an open problem. However he constructs two flag transitive planes [4] of order 25 and shows that his two planes of order 25 and the near field plane of order 9 have flag transitive collineation groups. C. Hering [5] has constructed a plane of order 27 which has a flag transitive group. Recently the author [6] has constructed a plane of order 49 and has shown that it has a flag transitive group. The aim of this paper is to construct a class of non-Desarguesian affine translation planes of order q^2, where q is a power of a prime $p\geqq 3$, which have flag transitive collineation groups.

2. Let $n=p^f$, where p is a prime and f is a positive integer. Let V be a vector space of dimension $2f$ over GF(p). Let $\{V_i|0\leqq i\leqq n\}$ be a set of f-dimensional subspaces of V. Let π be an incidence structure defined with vectors of V as points of π and subspaces V_i and their cosets (in the additive group of the vector space V) as lines of π with inclusion as an incidence relation. It may be shown (Andre [1]) that the incidence structure π is an affine (translation) plane if $V_i\cap V_j=\{0\}$, the subspace of V consisting of the zero vector alone, for $i\neq j$, $0\leqq i\leqq n$, $0\leqq j\leqq n$. Further any linear transformation of V, which permutes the subspaces V_i among themselves induces a collineation of π fixing the point corresponding to the zero

Received by the editors May 6, 1971 and, in revised form, August 31, 1972.

AMS (MOS) subject classifications (1970). Primary 20B25, 50D35; Secondary 20N10, 05B25.

Key words and phrases. Projective planes, affine planes, flag transitivity, Veblen-Wedderburn systems.

vector. It can be shown that π is flag transitive if there exists a group of linear transformations of V which permutes transitively the subspaces V_i for $0 \leqq i \leqq n$.

3. **Construction of a class of affine planes.** Let α be a generator of the group of nonzero elements of $GF(q^4)$, where q is a power of a prime $p \geqq 3$. Let β be the generator of the group of nonzero elements of $GF(q)$ given by $\beta = \alpha^{(q^2+1)(q+1)}$. Throughout this paper we use d in place of the number $(q+1)$. Since the element α^d lies outside $GF(q^2)$, it satisfies an equation

$$f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \tag{3.1}$$

where the coefficients a_i are from $GF(q)$ and the polynomial $f(x)$ is irreducible in $GF(q)$. Using the relations between the roots and the coefficients of equation (3.1) one may obtain the following:

$$a_0 = \beta^2, \tag{3.2}$$

$$a_1 = \beta a_3, \tag{3.3}$$

$$a_3 \neq 0, \tag{3.4}$$

$$a_2 + 2\beta + (\beta\alpha^{-d} + \alpha^d)a_3 + (\alpha^d - \beta\alpha^{-d})^2 = 0, \tag{3.5}$$

$$a_3 + (\alpha^d + \beta\alpha^{-d}) + (\alpha^d + \beta\alpha^{-d})^q = 0, \tag{3.6}$$

$$a_2 = 2\beta + (\beta\alpha^{-d} + \alpha^d)^d. \tag{3.7}$$

The relations (3.2), (3.3) and (3.7) are easy to verify. Using (3.2) and (3.3) in the relation

$$\alpha^{4d} + a_3\alpha^{3d} + a_2\alpha^{2d} + a_1\alpha^d + a_0 = 0 \tag{3.8}$$

we obtain (3.5). The relation (3.6) is a consequence of (3.5) and (3.7). To prove (3.4) let us suppose that $a_3 = 0$. Then (3.8) becomes

$$\alpha^{4d} + a_2\alpha^{2d} + \beta^2 = 0. \tag{3.8$'$}$$

The relation (3.8)′ implies that α^{2d} satisfies a quadratic in $GF(q)$, a contradiction since α^{2d} does not belong to $GF(q^2)$. Hence $a_3 \neq 0$.

LEMMA 3.1. *Let $u = (a_2 + 2\beta)a_3^{-1} + \beta\alpha^{-d} + \alpha^d$. Then $u \in GF(q^2)$ and is not a square in $GF(q^2)$. Consequently it may be expressed as $u = \alpha^{s(q^2+1)}$, where s is a certain odd integer.*

PROOF. From the relation

$$(a + b\alpha^d)^{q^2+1} = a^2 + b^2\beta + ab(\beta\alpha^{-d} + \alpha^d), \tag{3.9}$$

where $a, b \in GF(q)$ and $a \neq 0 \neq b$, we obtain that $(\beta\alpha^{-d} + \alpha^d)$ is an element in $GF(q^2)$. Consequently $u \in GF(q^2)$. Suppose that u is a square in $GF(q^2)$.

The relation (3.5) may now be written as

$$(3.10)\qquad (\alpha^d - \beta\alpha^{-d})^2 = -a_3 u.$$

Since any element of $\mathrm{GF}(q)$ is a square in $\mathrm{GF}(q^2)$, we obtain from (3.10) that $(\alpha^d-\beta\alpha^{-d})^2$ is a square in $\mathrm{GF}(q^2)$ and consequently $(\alpha^d-\beta\alpha^{-d}) \in \mathrm{GF}(q^2)$. This together with the fact that $(\beta\alpha^{-d}+\alpha^d) \in \mathrm{GF}(q^2)$ leads to a contradiction that $\alpha^d \in \mathrm{GF}(q^2)$. Thus $u \in \mathrm{GF}(q^2)$ and u is not a square in $\mathrm{GF}(q^2)$. Since u is not a square in $\mathrm{GF}(q^2)$, u may be expressed as $u=\alpha^{s(q^2+1)}$ where s is a certain odd integer.

LEMMA 3.2. *Let $v=(a_2+2\beta)^2a_3^{-2}-4\beta$. Then v is not a square in* $\mathrm{GF}(q)$.

PROOF. Let $(a_2+2\beta)a_3^{-1}=g$, $(\beta\alpha^{-d}+\alpha^d)=h$. We obtain from Lemma 3.1 that $\alpha^{s(q^2+1)}=(g+h)$ and therefore $(g+h)^d=\beta^s$ is not a square in $\mathrm{GF}(q)$, since s is an odd integer. However, using the relations (3.6) and (3.7) we obtain that

$$\begin{aligned}(g + h)^d &= (g + h^q)(g + h)\\ &= g^2 + g(h^q + h) + h^d\\ &= g^2 + a_2 - 2\beta + g(h^q + h)\\ &= g^2 - 4\beta + g(a_3 + h + h^q)\\ &= g^2 - 4\beta.\end{aligned}$$

Hence the lemma.

Let V_0 be the vector space over $\mathrm{GF}(q)$ defined by the basis $\{1, \alpha^d\}$. Let ν and δ be linear transformations of $\mathrm{GF}(q^4)$ defined by

$$\nu : x \to x\alpha^{2d} \quad \text{and} \quad \delta : x \to x^q\alpha^k$$

with $k \equiv s \pmod d$, where s is the odd integer of Lemma 3.1. Let π be the incidence structure whose points are the vectors of $V=\mathrm{GF}(q^4)$ and whose lines are the images of V_0 under the group $H=\langle \nu, \delta\rangle$ of linear transformations and their cosets in the additive group of $\mathrm{GF}(q^4)$, with inclusion as an incidence relation.

THEOREM 3.1. *The incidence structure π is a non-Desarguesian affine translation plane. Further the group H of linear transformations induces a group of collineations of π which fixes the origin and permutes the lines through the origin transitively.*

In the course of the proof of Theorem 3.1 we need the following two lemmas.

Let $0\neq x=a+b\alpha^d$ and $y=(b+\beta^{-1}a\alpha^d)$ be elements from V_0, where $a, b \in \mathrm{GF}(q)$. From the relation

$$(3.11)\qquad (xy^{-1})^{q^2+1} = \beta$$

we obtain that

$$xy^{-1} = \alpha^{d+t(q^2-1)} = \alpha^{k_x d} \tag{3.12}$$

for some integer t and therefore k_x is an odd integer, a function of x.

LEMMA 3.3. *Let $0\neq x=a+b\alpha^d$, $y=(b+\beta^{-1}a\alpha^d)$ and z be elements of V_0 where a, $b\in \mathrm{GF}(q)$ and $xz^{-1}\notin \mathrm{GF}(q)$. Then $x=z\alpha^{cd}$ for some integer c if and only if* (i) *$z=ly$ for some $l\in \mathrm{GF}(q)$ and* (ii) *$l\alpha^{cd}=\alpha^{k_x d}$. Further if $x=z\alpha^{cd}$, then c is an odd integer.*

PROOF. Obviously (i) and (ii) imply that $x=z\alpha^{cd}$ for some integer c and $z=e+f\alpha^d$. Suppose that $x=z\alpha^{cd}$ for some integer c and $z=e+f\alpha^d$. Obviously $z\neq 0$. From the relation $x^{q^2+1}=z^{q^2+1}\beta^c$ we obtain, after using the fact that 1, α^{-d}, α^d are linearly independent over $\mathrm{GF}(q)$,

$$ab = ef\beta^c, \tag{3.13}$$

$$a^2+b^2\beta = (e^2+f^2\beta)\beta^c. \tag{3.14}$$

Eliminating β^c from (3.13) and (3.14) we have

$$(\beta bf - ae)(be-af) = 0. \tag{3.15}$$

Since $xz^{-1}\notin \mathrm{GF}(q)$, $be-af\neq 0$. We therefore have that $\beta bf-ae=0$ from which we obtain that $z=l(b+\beta^{-1}a\alpha^d)$ for some $l\in\mathrm{GF}(q)$. The condition (ii) now follows easily. Since $l\alpha^{cd}=\alpha^{k_x d}$, where k_x is an odd integer, we have that c also is an odd integer.

LEMMA 3.4. *Let $M=\{xy\alpha^{ld} \mid x, y\in V_0, x\neq 0\neq y, l$ an integer$\}$. Then $\alpha^m\notin M$ where $m\equiv s$ (mod d) and $\alpha^{s(q^2+1)}=u$ of Lemma* 3.1.

PROOF. Let $x=a+b\alpha^d$ and $y=e+f\alpha^d$, where a, b, e, $f\in\mathrm{GF}(q)$. Suppose that $xy\alpha^{ld}=\alpha^{s+td}$ for some integers l and t. Then

$$(xy)^{q^2+1}\beta^{l-t} = u \tag{3.16}$$

using the relations (3.2), (3.3) and (3.8) in (3.16) we obtain

$$\begin{aligned}\beta^{-t+l}((\beta bf+ae)^2+\beta(b^2e^2+a^2f^2)-abefa_2)&\\ +\beta^{l-t}((\beta bf+ae)(be+af)-abefa_3)(\beta\alpha^{-d}+\alpha^d)&\\ =(a_2+2\beta)a_3^{-1}+\beta\alpha^{-d}+\alpha^d.&\end{aligned} \tag{3.17}$$

Since 1 and $(\beta\alpha^{-d}+\alpha^d)$ are linearly independent over $\mathrm{GF}(q)$ we obtain, from (3.17),

$$\beta^{l-t}((\beta bf+ae)^2+\beta(b^2e^2+a^2f^2)-abefa_2) = (a_2+2\beta)a_3^{-1}, \tag{3.18}$$

$$\beta^{l-t}((\beta bf+ae)(be+af)-abefa_3) = 1. \tag{3.19}$$

Eliminating β^{l-t} from (3.18) and (3.19) we obtain

$$(\beta bf + ae)^2 - (a_2 + 2\beta)a_3^{-1}(\beta bf + ae)(be + af) + \beta(be + af)^2 = 0. \tag{3.20}$$

Suppose $be+af=0$. Then $\beta bf+ae=0$. If we further suppose that any one of a, b, e, f vanishes, then we obtain that either $x=0$ or $y=0$, contrary to the hypothesis. Thus in case $be+af=0$, we have $a\neq 0\neq b$, $e\neq 0\neq f$. Eliminating a and b from $be+af=0$ and $\beta bf+ae=0$ we obtain $\beta=e^2f^{-2}$, a square in GF(q), a contradiction. Thus $be+af\neq 0$. Equation (3.20) may now be written as

$$w^2 - (a_2 + 2\beta)a_3^{-1}w + \beta = 0, \tag{3.21}$$

where $w=(\beta bf+ae)(be+af)^{-1}$. Since $((a_2+2\beta)^2a_3^{-2}-4\beta)$ is not a square in GF(q), the relation (3.21) leads to a contradiction that w satisfies a quadratic irreducible in GF(q). From this contradiction we infer the truth of the lemma.

PROOF OF THEOREM 3.1. Let V_i be the vector space over GF(q) generated by $\{\alpha^{2id}, \alpha^{(2i+1)d}\}$ for $0\leqq i\leqq (q^2-1)/2$. Obviously V_i is the image of V_0 under the linear transformation ν^i of GF(q^4). Let U_i be the vector space over GF(q) generated by $\{\alpha^{k+d(q+2i)}, \alpha^{k+2id}\}$ for $0\leqq i\leqq (q^2-1)/2$. As before it may be shown that U_0 is the image of V_0 under the linear transformation δ and U_i is the image of U_0 under the linear transformation ν^i of GF(q^4). Further from the relation

$$(a\alpha^{2id} + b\alpha^{(2i+1)d})\delta = \alpha^{2iqd}(a\alpha^k + b\alpha^{k+qd}),$$

where $a, b\in$ GF(q), we obtain that $V_i\delta=U_j$ where $iq\equiv j(\mathrm{mod}(q^2+1)/2)$. Similarly from the relation

$$(a\alpha^{k+2id} + b\alpha^{k+(2i+q)d})\delta = \alpha^{(2iqd)+(k-1)d}(b\beta + a\alpha^d)$$

we obtain that $U_i\delta=V_j$, where $iq+(k-1)/2\equiv j$ $(\mathrm{mod}(q^{2+1})/2)$. Thus the set P of images of V_0 under the group $H=\langle\nu, \delta\rangle$ of linear transformations of GF(q^4) consists of V_i and U_j for $0\leqq i\leqq (q^2-1)/2$ and $0\leqq j\leqq (q^2-1)/2$ and H is transitive on the set P. We may now conclude that, if π is an affine plane, then H induces a collineation group which fixes the origin and permutes the lines through the origin transitively.

To prove that π is an affine plane we have to show that $X_i\cap Y_j=\{0\}$ if $X_i\neq Y_j$ and $X=U$ or V and $Y=U$ or V, $0\leqq i\leqq (q^2-1)/2$, $0\leqq j\leqq (q^2-1)/2$. Without loss of generality we may suppose $i\leqq j$. Then from the relations $(X_i\cap Y_j)\nu^{-i}=X_0\cap Y_{j-i}$ and $(U_i\cap U_j)\delta^{-1}=V_0\cap V_{j-i}$ we have that

$$X_i \cap Y_j = \{0\} \quad \text{if and only if } X_0 \cap Y_{j-i} = \{0\}, \tag{3.22}$$

$$U_i \cap U_j = \{0\} \quad \text{if and only if } V_0 \cap V_{j-i} = \{0\}. \tag{3.23}$$

In view of (3.22) and (3.23) it is enough if we show that

(3.24) $$V_0 \cap V_i = \{0\} \quad \text{for } 0 \leqq i \leqq (q^2 - 1)/2,$$

(3.25) $$V_0 \cap U_i = \{0\} \quad \text{for } 0 \leqq i \leqq (q^2 - 1)/2.$$

Obviously V_0, V_i, U_j contain the zero vector. To prove (3.24) let us suppose $x \in V_0$, $y \in V_i$, with $x \neq 0 \neq y$ and $x=y$. Then there is a $z \in V_0$ such that $y=z\alpha^{2id}$ and consequently $x=z\alpha^{2id}$. This in view of Lemma 3.3 leads to a contradiction that $2i$ is an odd integer, since $xz^{-1} \notin \mathrm{GF}(q)$. Suppose $0 \neq x \in V_0$ and $0 \neq y \in U_i$ and $x=y$. Then there is a $z \neq 0$ in U_0 such that $y=z\alpha^{2id}$. Let $z=(e+f\alpha^{qd})\alpha^k$ where $e, f \in \mathrm{GF}(q)$. Then $e+f\alpha^{qd}=(e+f\alpha^d)^q=(e+f\alpha^d)^{-1}(e+f\alpha^d)^d$. Let $e+f\alpha^d=\alpha^t$. Now $x=y$ implies that $x(e+f\alpha^d)=\alpha^{k+td+2id}$ contrary to Lemma 3.4. Thus π is an affine plane.

Obviously $A=\langle\alpha^{2d}\rangle$ induces a group of collineations of π and its order is $(q^2+1)(q-1)/2$. Suppose T is an odd prime which divides (q^4-1) but does not divide (p^i-1) for $0<i<4r$ (such a prime exists in view of Corollary 2, p. 358 of Artin [**2**]). Obviously T is not a factor of $2(q+1)$. Otherwise we obtain a contradiction that $q^2 \equiv 1 (\mathrm{mod}\ T)$. It then follows that T is a factor of $(q^2+1)(q-1)/2$, the order of A and satisfies conditions (2) and (3) of Lemma 3.1 of Foulser [**4**]. We now claim that V_0 is not of the form $A(\mathrm{GF}(q^2))$ for any $a \neq 0$ from $\mathrm{GF}(q^4)$. Suppose the contrary. Then if $b \neq 0$ and bV_0, it follows that $b^{-1}V_0=\mathrm{GF}(q^2)$ and it may be shown that it is not the case by taking $b=\alpha^d$ and noting that $\alpha^{-d} \notin \mathrm{GF}(q^2)$. We now invoke Lemma 6.1 of Foulser [**4**] to claim that π is non-Desarguesian. This completes the proof of the theorem.

Classification of these planes into nonisomorphic classes will be discussed elsewhere.

References

1. J. André, *Über nicht-Desarguesche Ebenen mit transitiver Translationsgruppe*, Math. Z. **60** (1954), 156–186. MR **16**, 64.

2. E. Artin, *The orders of linear groups*, Comm. Pure Appl. Math. **8** (1955), 355–366. MR **17**, 12.

3. D. A. Foulser, *The flag-transitive collineation groups of the finite Desarguesian affine planes*, Canad. J. Math. **16** (1964), 443–472. MR **29** #3549.

4. ———, *Solvable flag transitive affine groups*, Math. Z. **86** (1964), 191–204. MR **30** #1190.

5. C. Hering, *Eine nicht-desarguesche zweifach transitive affine Ebene der Ordnung* 27, Abh. Math. Sem. Univ. Hamburg **34** (1969/70), 203–208. MR **42** #8390.

6. M. L. Narayana Rao, *A flag transitive plane of order* 49, Proc. Amer. Math. Soc. **32** (1972), 256–262.

7. A. Wagner, *On finite affine line transitive planes*, Math. Z. **87** (1965), 1–11. MR **30** #2391.

Department of Mathematics, Osmania University, Hyderabad-7 (A.P.), India